SYNOPSIS

OF

LINEAR ASSOCIATIVE ALGEBRA

A REPORT ON ITS NATURAL DEVELOPMENT AND RESULTS REACHED
UP TO THE PRESENT TIME

BY

JAMES BYRNIE SHAW

Professor of Mathematics in the James Milliken University

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CONTENTS.

ERRATA.

Page.

11. Line 13, *for* $|e_{ij}|^2$ *read* $|c_{ij}|^2$.

15. In the foot-notes change numbering as follows: *for* 1 *read* 2, *for* 2 *read* 3, *for* 3 *read* 4, *for* 4 *read* 1.

26. Line 21, *for* h^{i1} *read* h_{i1}.

33. Line 15, *for* $A e_j e_k$ *read* $A e_j e_i$.

34. Line 6, *for* $[m_1(\rho_1)$ *read* $[m_1'(\rho_1)$.

49. Line 6, *for* $m_{i+1}^{(i)}$ *read* m'_{i+1}.

53, 54. In the table for $r > 6$ in every instance *change* $r-2$ to $r-3$, and $r-3$ to $r-4$. *In case* (27), *however, read* $e_2 = (211) - (12\,r-3)$.

57. Line 8, *for* t_i *read* t_i.

59. Line 33, *remove* the period after A.

67. Line 12, *insert* a comma (,) after "integer".

68. Lines 9 and 10, *change* γ to ν.

71. Line 17, in type III *for* e_{i1} *read* e_{1i}.

72. Last line, *for* $a\,q\,a^{-1}$ *read* $a\,q\,a^{-1}$.

73. Line 3 from bottom, *for* $jk^{\frac{2n}{5}}$ *read* $jk^{\frac{2s}{5}}$.

94. Line 7, *for* Si^2j' *read* $Si^2j'^2$.

94. Last line, in the second column of the determinant and third line *for* $S.j^{-1}\alpha j^2\phi - \phi$ *read* $S.j^{-1}\alpha j^2\phi$.

100. Line 12, *for* $\phi = \phi$ *read* $\phi = \breve{\phi}$.

106. Some of these cases are equivalent to others previously given.

107. Line 3 from bottom, *for* $e_2 = (221)$ *read* $e_2 = (211)$.

116. Line 25 *should read* $p = \dfrac{x - 2\,V.\,a\overline{b}}{f}$.

124. Note 3, *add*: cf. Beez 2.

128. Line 11, *for* $i = 1 \ldots . k_x$ *read* $i = 1 \ldots . h_x$.

SYNOPSIS OF LINEAR ASSOCIATIVE ALGEBRA

INTRODUCTION.[1]

This memoir is genetic in its intent, in that it aims to set forth the present state of the mathematical discipline indicated by its title: not in a comparative study of different known algebras, nor in the exhaustive study of any particular algebra, but in tracing the general laws of the whole subject. Developments of individual known algebras may be found in the original memoirs. A partial bibliography of this entire field may be found in the Bibliography of the Quaternion Society,[2] which is fairly complete on the subject. Comparative studies, more or less complete, may be found in HANKEL'S lectures,[3] and in CAYLEY'S paper on Multiple Algebra.[4] These studies, as well as those mentioned below, are historical and critical, as well as comparative. The phyletic development is given partially in STUDY'S Encyklopädie[5] article, his *Chicago Congress*[6] paper, and in CARTAN'S Encyclopédie[7] article. These papers furnish numerous expositions of systems, and references to original sources. Further historical references are also indicated below.[8]

In view of this careful work therefore, it does not seem desirable to review the field again historically. There is a necessity, however, for a presentation of the subject which sets forth the results already at hand, in a genetic order. From such presentation may possibly come suggestions for the future. Attention will be given to chronology, and it is hoped the references given will indicate priority claims to a certain extent. These are not always easy to settle, as they are sometimes buried in papers never widely circulated, nor is it always possible to say whether a notion existed in a paper explicitly or only implicitly, consequently this memoir does not presume to offer any authoritative statements as to priority.

The memoir is divided into three parts: *General Theory*, *Particular Systems*, *Applications*. Under the *General Theory* is given the development of the subject from fundamental principles, no use being made of other mathematical disciplines, such as bilinear forms, matrices, continuous groups, and the like.

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[1] Presented, in a slightly different form, as an abstract of this paper, to the Congress of Arts and Sciences at the Universal Exposition, St. Louis, Sept. 22, 1904.

[2] Bibliography of Quaternions and allied systems of mathematics, Alexander Macfarlane, 1904, Dublin.

[3] HANKEL 1. References to the bibliography at the end of the memoir are given by author and number of paper.

[4] CAYLEY 9. [5] STUDY 8. [6] STUDY 7. [7] CARTAN 3.

[8] BEMAN 2, GIBBS 2, R. GRAVES 1, HAGEN 1, MACFARLANE 4.

We find the first such general treatment in HAMILTON'S theory[1] of sets. The first extensive attempt at development of algebras in this way was made by BENJAMIN PEIRCE[2]. His memoir was really epoch-making. It has been critically examined by HAWKES[3], who has undertaken to extend Peirce's method, showing its full power[4]. The next treatment of a similar character was by CARTAN[5], who used the characteristic equation to develop several theorems of much generality. In this development appear the *semi-simple*, or Dedekind, and the *pseudo-nul*, or nilpotent, sub-algebras. The very important theorem that the structure of every algebra may be represented by the use of double units, the first factor being quadrate, the second non-quadrate, is the ultimate proposition he reaches. The latest direct treatment is by TABER[6], who reëxamines the results of PEIRCE, establishing them fully (which Peirce had not done in every case) and extending them to any domain for the coordinates. [His units however are linearly independent not only in the field of the coordinates, but for any domain or field]

Two lines of development of linear associative algebra have been followed besides this direct line. The first is by use of the continuous group. It was POINCARÉ[7] who first announced this isomorphism. The method was followed by SCHEFFERS[8], who classified algebras as quaternionic and non-quaternionic. In the latter class he found "regular" units which can be so arranged that the product of any two is expressible linearly in terms of those which follow both. He worked out complete lists of all algebras to order five inclusive. His successor was MOLIEN[9], who added the theorems that quaternionic algebras contain independent quadrates, and that quaternionic algebras can be classified according to non-quaternionic types. He did not, however, reach the duplex character of the units found by CARTAN.

The other line of development is by using the matrix theory. C. S. PEIRCE[10] first noticed this isomorphism, although in embryo it appeared sooner. The line was followed by SHAW[11] and FROBENIUS[12]. The former shows that the equation of an algebra determines its quadrate units, and certain of the *direct* units; that the other units form a nilpotent system which with the quadrates may be reduced to certain canonical forms. The algebra is thus made a sub-algebra under the algebra of the *associative units* used in these canonical forms. FROBENIUS proves that every algebra has a DEDEKIND sub-algebra, whose equation contains all factors in the equation of the algebra. This is the semi-simple algebra of CARTAN. He also showed that the remaining units form a nilpotent algebra whose units may be *regularized.*

It is interesting to note the substantial identity of these developments, aside from the vehicle of expression. The results will be given in the order of development of the paper with no regard to the method of derivation. The references will cover the different proofs.

[1] HAMILTON 1. [2] B. PEIRCE 1, 3. [3] HAWKES 2. [4] HAWKES 1, 3, 4.
[5] CARTAN 2. [6] TABER 4. [7] POINCARÉ 1. [8] SCHEFFERS 1, 2, 3.
[9] MOLIEN 1. [10] C. S. PEIRCE 1, 4. [11] SHAW 4. [12] FROBENIUS 14.

The last chapter of the general theory gives a sketch of the theory of general algebra, placing linear associative algebra in its genetic relations to general linear algebra. Some scant work has been done in this development, particularly along the line of symbolic logic.[1] On the philosophical side, which this general treatment leads up to, there have always been two views of complex algebra. The one regards a number in such an algebra as in reality a duplex, triplex, or multiplex of arithmetical numbers or expressions. The so-called units become mere *umbrae* serving to distinguish the different coordinates. This seems to have been CAYLEY'S[2] view. It is in essence the view of most writers on the subject. The other regards the number in a linear algebra as a single entity, and multiplex only in that an equality between two such numbers implies n equalities between certain coordinates or functions of the numbers. This was HAMILTON'S[3] view, and to a certain extent GRASSMANN'S.[4] The first view seeks to derive all properties from a multiplication table. The second seeks to derive these properties from definitions applying to all numbers of an algebra. The attempt to base all mathematics on arithmetic leads to the first view. The attempt to base all mathematics on algebra, or the theory of entities defined by relational identities, leads to the second view. It would seem that the latter would be the more profitable from the standpoint of utility. This has been the case notably in all developments along this line, for example, quaternions and space-analysis in general. HAMILTON, and those who have caught his idea since, have endeavored to form expressions for other algebras which will serve the purpose which the scalar, vector, conjugate, etc., do in quaternions, in relieving the system of reference to any unit-system. Such definition of algebra, or of an algebra, is a development in terms of what may be called the fundamental invariant forms of the algebra. The characteristic equation of the algebra and its derived equations are of this character, since they are true for all numbers irrespective of the units which define the algebra; or, in other words, these relations are identically the same for all *equivalent* algebras. The present memoir undertakes to add to the development of this view of the subject.

In conclusion it may be remarked that several theorems occur in the course of the memoir which it is believed have never before been explicitly stated. Where not perfectly obvious the proof is given. The proofs of the known theorems are all indicated by the references given, the papers referred to containing the proofs in question. No fuller treatment could properly be given in a synopsis.

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[1] C. S. PEIRCE 1, 2, SCHROEDER 1, WHITEHEAD 1, RUSSELL 1, SHAW 1.

[2] CAYLEY 1, 9. See also GIBBS 1, 2, 3. [3] HAMILTON 1, 2. [4] GRASSMANN 1, 2.

PART I. GENERAL THEORY.

I. DEFINITIONS.

1. EARLY DEFINITIONS.[1]

1. Definitions. Let there be a set of r entities, $e_i \ldots . e_r$, which will be called *qualitative units*. These entities will serve to distinguish certain other entities, called *coordinates*, from each other, the coordinates belonging to a given *range*, or ensemble of elements; thus if a_i is a coordinate, then $a_i e_i$ is different from $a_i e_j$, if $i \neq j$, and no process of combination belonging to the range of a_i can produce $a_i e_j$ from $a_i e_i$. Thus, the range may be the domain of scalars (ordinary, real, and imaginary numbers), or it may be the range of integers, or it may be any abstract field, or even any algebra. If it be the range of integers, subject to addition, subtraction, multiplication, and partially to division, then by no process of this kind or any combination of such can $a_i e_i$ become $a_i e_j$. These *qualified coordinates* may be combined into expressions called *complex*, or *hypercomplex*, or *multiple numbers*, thus

$$\alpha = \sum_{i=1}^{r} a_i e_i$$

In this number each a_i is supposed to run through the entire range. The units e_i, or $1 e_i$, are said to define a *region* of order r.

2. Theorems:[2]

(1) $(a+b) e_i = a e_i + b e_i$, and conversely, if $+$ is defined for the range.

(2) $\quad 0 e_i = e_i 0 = 0$, if 0 belongs to the range.

(3) $\sum_{i=1}^{r} a_i e_i = 0$, implies $a_i = 0 \qquad (i = 1 \ldots . r)$

(4) If $\sum_{i=1}^{r} a_i e_i = \sum_{i=1}^{r} b_i e_i$; then $a_i = b_i$, $i = 1 \ldots . r$, and conversely.

Theorems (3) and (4) might be omitted by changing the original definitions, in which case relations might exist between the units. Thus, the units $+1$ and -1 are connected by the relation $+1 + (-1) = 0$.

Algebras of this character have more units than dimensions.

3. Definitions. A combination of these multiple numbers called *addition* is defined by the statement

$$\alpha + \beta = \sum_{i=1}^{r} (a_i + b_i) e_i$$

---

[1] HANKEL 1, WHITEHEAD 1. Almost every writer has given equivalent definitions. These were of course more or less loosely stated.

[2] WHITEHEAD 1.

In quaternions and space-analysis the definition is derived from geometrical considerations, and the definition used here is usually a theorem.[1]

4. **Theorem.** From the definition we have

$$\alpha + \beta = \beta + \alpha \qquad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

when these equations hold for the range of coordinates. If subtraction is defined for the range, it will also apply here.

5. **Theorem.** If m belongs to the range and if $m\,a$ is defined for the range (called multiplication of elements of the range) then we have

$$m\,\alpha = \sum_{i=1}^{r} (m\,a_i)\,e_i$$

6. The units are called *units*[2], or *Haupteinheiten*[3], and the *region* they define is also called the *ground*[4] or the *basis*[5] of the algebra. The units are written also[6] $(1, 0, 0, \ldots.), (0, 1, 0, \ldots.), \ldots.\ (0, 0, \ldots. 1)$, the position of the 1 serving to designate them. The implication in this method of indicating them is that they are simply ordinary units (numbers) in a system of n-tuple numbers, the coordinates of each n-tuple number being independent variables. This view may be called the *arithmetic* view as opposed to that which may be called the *vector* view, and which looks upon the units as *extraordinary* entities, a term due to CAYLEY. There are two other views of the units, namely, the *operator* view, and the *algebraic* view. The first considers any unit except ordinary unity to be an operator, as (-1) or the quaternions i, j, k. The second considers any unit to be a solution of a set of equations which it must satisfy and as an extension of some range (or domain, or field); or from a more abstract point of view we consider the range to be reduced modulo certain expressions containing the so-called units as arbitrary entities from the range. Thus, if we treat algebraic expressions modulo $i^2 + 1$, we virtually introduce $\sqrt{-1}$ into the range as an extension of it.[7]

7. **Definition.** We may now build a calculus[8] based solely on addition of numbers and combinations of the coordinates. This may be done as follows:

Let the symbol I have the meaning defined by the following equations: if

$$\alpha = \sum_{i=1}^{r} a_i e_i \qquad \xi = \sum_{i=1}^{r} x_i e_i$$

then

$$I\,.\,\alpha\,\xi = \sum_{i=1}^{r} a_i\,x_i$$

It is assumed that the coordinates a, x, are capable of combining by an associative, commutative, distributive process which may be called multiplication, so that $a_i x_i$ is in the coordinate range for every a_i and x_i, as well as $\Sigma a_i x_i$.

[1] HAMILTON 1, 2, GRASSMANN 1, 2, cf. MACFARLANE 1. [2] GRASSMANN 1. [3] WEIERSTRASS 2. [4] TABER 1. [5] MOLIEN 1. [6] DEDEKIND 1, BERLOTY 1. [7] SHAW 13. [8] See § 21 for difference between a *calculus* and an *algebra*.

Evidently
$$I \,.\, e_i \xi = x_i \qquad \xi = \sum_{i=1}^{r} e_i I \,.\, e_i \xi_i$$
Also, if $i \neq j$
$$I \,.\, e_i e_i = 1 \qquad I \,.\, e_i e_j = 0$$

8. Theorem. We have
$$I \,.\, \alpha\xi = I \,.\, \xi\alpha \qquad I \,.\, \xi\xi = \Sigma \,.\, x_i^2$$

9. Definition. We say that α and ξ are *orthogonal* if $I \,.\, \alpha\xi = 0$. The units $e_1 \ldots, e_r$ therefore form an *orthogonal* system.

If $I \,.\, \xi\xi = 0$, ξ is called a *nullitat.*

10. Theorem. Let
$$E_i = \sum_{j=1}^{r} c_{ij} e_j \qquad (i = 1 \ldots r)$$
and
$$I \,.\, E_i E_j = 0,\ i \neq j \qquad I \,.\, E_i E_i = 1 \qquad |e_{ij}|^2 = 1$$
Then, we have
$$e_j = \frac{\Sigma \,.\, C_{ij} E_i}{|c_{ij}|}$$
where C_{ij} is the minor of c_{ij} in $|c_{ij}|$.

Further
$$\xi = \frac{\Sigma x_j C_{ij} E_i}{|c_{ij}|} \qquad \alpha = \frac{\Sigma a_k C_{ik} E_i}{|c_{ij}|}$$
If I'. refers to the E coordinates just as I to those of the e's,
$$I' . \xi\alpha = \frac{\Sigma x_j a_k C_{ij} C_{ik}}{|c_{ij}|^2} = I \,.\, \xi\alpha$$
since $\sum_{i=1}^{r} C_{ij} C_{ik} = 0$ or 1 as $k \neq j$ or $k = j$, and $|c_{ij}|^2 = 1$.

Hence I is invariant under a change to a new orthogonal basis.

11. Definition. Let the expression $A \,.\, \alpha_1 \ldots \alpha_{m-1} A\beta_1 \ldots \beta_m$ represent the determinant
$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \ldots\ldots & \beta_m \\ I\alpha_1\beta_1 & I\alpha_1\beta_2 & I\alpha_1\beta_3 \ldots\ldots & I\alpha_1\beta_m \\ I\alpha_2\beta_1 & I\alpha_2\beta_2 & I\alpha_2\beta_3 \ldots\ldots & I\alpha_2\beta_m \\ \ldots & \ldots & \ldots & \ldots \\ I\alpha_{m-1}\beta_1 & I\alpha_{m-1}\beta_2 & I\alpha_{m-1}\beta_3 \ldots & I\alpha_{m-1}\beta_m \end{vmatrix}$$
In particular
$$A \,.\, \alpha_1 A\beta_1\beta_2 = \beta_1 I\alpha_1\beta_2 - \beta_2 I\alpha_1\beta_1$$
$$I \,.\, \alpha_1 A\alpha_2 A\beta_1\beta_2 = |I\alpha_1\beta_1,\ I\alpha_2\beta_2| = -I \,.\, \alpha_2 A\alpha_1 A\beta_1\beta_2$$
$$= I .\ \beta_1 A\beta_2\, A\alpha_1\alpha_2$$
These expressions vanish if $\alpha_1 \ldots \alpha_{m-1}$ are connected by any linear relation, or $\beta_1 \ldots \beta_m$ by any linear relation, or if any α is orthogonal to all of the β's. If any β, say β_1, is orthogonal to all the α's,
$$I \,.\, \alpha_1 A\alpha_2 \ldots \alpha_m A\beta_1 \ldots \beta_m = 0$$
and
$$A\alpha_1 \ldots \alpha_{m-1} A\beta_1 \ldots \beta_m = \beta_1 I\alpha_1 A\alpha_2 \ldots \alpha_{m-1} A\beta_2 \ldots \beta_m$$

12. Theorem.

$$A\,.\,\alpha A\beta\gamma + A\,.\,\beta A\gamma\alpha + A\,.\,\gamma A\alpha\beta = 0$$

$$I\,.\,\alpha A\beta A\gamma\delta + I\,.\,\beta A\gamma A\alpha\delta + I\,.\,\gamma A\alpha A\beta\delta = 0$$

$$\beta I\,\alpha\alpha = \alpha I\alpha\beta - A\alpha A\alpha\beta$$

$$A\,.\,\alpha_1\alpha_2 A\beta_1\beta_2\beta_3 = \beta_1 I\,.\,\alpha_1 A\alpha_2 I\beta_2\beta_3 - \beta_2 I\,.\,\alpha_1 A\alpha_2 A\beta_1\beta_3 + \beta_3 I\,.\,\alpha_1 A\alpha_2 A\beta_1\beta_2$$
$$= A\alpha_1 A\beta_1\beta_2 I\alpha_2\beta_3 - A\,\alpha_1 A\beta_1\beta_3 I\alpha_2\beta_2 + A\,\alpha_1 A\beta_2\beta_3 I\alpha_2\beta_1$$

$$I\,.\,\alpha_1 A\alpha_2\alpha_3 A\beta_1\beta_2\beta_3 = -I\,.\,\alpha_2 A\alpha_1\alpha_3 A\beta_1\beta_2\beta_3 = \ldots\ldots$$
$$= I\,.\,\beta_1 A\beta_2\beta_3 A\alpha_1\alpha_2\alpha_3 = -I\,.\,\beta_2 A\beta_1\beta_3 A\alpha_1\alpha_2\alpha_3 = \ldots\ldots$$

$$\gamma I\,.\,\alpha A\beta A\alpha\beta = \alpha I\,.\,\alpha A\beta A\gamma\beta + \beta I\,.\,\alpha A\beta A\alpha\gamma + A\alpha\beta A\alpha\beta\gamma$$

13. Theorem. In general

$$A\,.\,\alpha_1\ldots\alpha_{m-1}A\beta_1\ldots\beta_m = \Sigma\,.\,\beta_1 I\alpha_1 A\alpha_2\ldots\alpha_{m-1}A\beta_2\ldots\beta_m$$
$$= \Sigma\,.\,A\alpha_1 A\beta_1\beta_2 I\alpha_2 A\alpha_3\ldots\alpha_{m-1}A\beta_3\ldots\beta_m$$
$$= \Sigma\,.\,A\alpha_1\alpha_2 A\beta_1\beta_2\beta_3 I\,.\,\alpha_3 A\alpha_4\ldots\alpha_{m-1}A\beta_4\ldots\beta_m$$
$$= \ldots\ldots$$
$$= \Sigma\,.\,A\,.\,\alpha_1\ldots\alpha_{m-2}A\beta_1\ldots\beta_{m-1}\,.\,I\alpha_{m-1}\beta_m$$

$$\beta I\,.\,\alpha_1 A\alpha_2\ldots\alpha_{m-1}A\alpha_1\ldots\alpha_{m-1} = \Sigma\,.\,\alpha_1 I\alpha_1 A\alpha_2\ldots\alpha_{m-1}A\beta\alpha_2\ldots\alpha_{m-1}$$
$$+ (-)^{m-1}A\,.\,\alpha_1\ldots\alpha_{m-1}A\alpha_1\ldots\alpha_{m-1}\beta$$

Signs of terms follow rule for Laplace's expansion of a determinant. Developments for $A\alpha_1 A\beta_1\gamma$ and higher forms are easily found.

14. Theorem. If the notation be used

$$A\begin{Bmatrix}\mu_{12\ldots s}\\ \lambda_{12\ldots s}\end{Bmatrix} = A\,.\,\lambda_1 A\,.\,A\lambda_2\ldots A\,.\,A\lambda_s A\mu_0\mu_s\,.\,\mu_{s-1}\ldots\mu_2\,.\,\mu_1$$

then

$$A\begin{Bmatrix}\mu_{12\ldots s}\\ \lambda_{12\ldots s}\end{Bmatrix} = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3\ldots & \mu_{s-1} & \mu_s \\ I\lambda_1\mu_0 & I\lambda_1\mu_1 & I\lambda_1\mu_2 & I\lambda_1\mu_3\ldots & I\lambda_1\mu_{s-1} & I\lambda_1\mu_s \\ I\lambda_2\mu_0 & 0 & I\lambda_2\mu_2 & I\lambda_2\mu_3\ldots & I\lambda_2\mu_{s-1} & I\lambda_2\mu_s \\ I\lambda_3\mu_0 & 0 & 0 & I\lambda_3\mu_3\ldots & I\lambda_3\mu_{s-1} & I\lambda_3\mu_s \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ I\lambda_s\mu_0 & 0 & 0 & 0\ \ldots & 0 & I\lambda_s\mu_s \end{vmatrix}\,.$$

$$= \mu_0 I\lambda_1\mu_1 I\lambda_2\mu_2\ldots I\lambda_s\mu_s$$
$$- \mu_s I\lambda_1\mu_1\ldots I\lambda_{s-1}\mu_{s-1} I\lambda_s\mu_0$$
$$- \mu_{s-1} I\lambda_1\mu_1\ldots I\lambda_{s-2}\mu_{s-2} I\lambda_{s-1} A\begin{Bmatrix}\mu_s\\ \lambda_s\end{Bmatrix}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$- \mu_1 I\lambda_1 A\begin{Bmatrix}\mu_{2\ldots s}\\ \lambda_{2\ldots s}\end{Bmatrix}$$

It follows that

$$A\,.\,\lambda_1\lambda_2 A\,\mu_0\mu_1\mu_2 = A\begin{Bmatrix}\mu_{12}\\ \lambda_{12}\end{Bmatrix} - I\,.\,\lambda_2\mu_1\,.\,A\begin{Bmatrix}\mu_2\\ \lambda_1\end{Bmatrix}$$

Omitting λ and μ

$$A\,.\,\lambda_1\lambda_2\lambda_3 A\,\mu_0\mu_1\mu_2\mu_3 = A\begin{Bmatrix}123\\123\end{Bmatrix} - I\lambda_2\mu_1 A\begin{Bmatrix}23\\13\end{Bmatrix} - I\lambda_3\mu_1 A\begin{Bmatrix}23\\21\end{Bmatrix}$$
$$- I\lambda_3\mu_2 A\begin{Bmatrix}13\\12\end{Bmatrix} + I\lambda_3\mu_2 I\lambda_2\mu_1 A\begin{Bmatrix}3\\1\end{Bmatrix} + I\lambda_3\mu_1 I\lambda_1\mu_2 A\begin{Bmatrix}3\\2\end{Bmatrix}$$

The forms $A \ldots\ldots A \ldots\ldots$ may all be developed in this manner.

The form $A\begin{Bmatrix}i_1 i_2 \ldots\ldots i_n\\ j_1 j_2 \ldots\ldots j_n\end{Bmatrix}$, where $i_1 \ldots\ldots i_n$, $j_1 \ldots\ldots j_n$ are two sets of n subscripts each chosen from among the r numbers $1 \ldots\ldots r$, may be looked upon as determining a substitution of n cycles on the r numbers, the multipliers $I\lambda_{j_{n+1}}\mu_{i_{n+1}}, \ldots\ldots I\lambda_{j_r}\mu_{i_r}$ furnishing the other $r-n$ numbers, that is, the whole term determines the substitution

$$\begin{Bmatrix}i_{n+1} \ldots\ldots i_r, i_1 \ldots\ldots i_n\\ j_{n+1} \ldots\ldots j_r, j_1 \ldots\ldots j_n\end{Bmatrix}$$

which must contain just n cycles. It is also to be noticed that $i_t \gtreqless j_t$, $t = 1 \ldots\ldots n$. The terms in the expansion of $A\,.\,\lambda_1 \ldots\ldots \lambda_r A\mu_0\mu_1 \ldots\ldots \mu_r$ are then the $r!$ terms corresponding to the $r!$ substitutions of the symmetric group of order $r!$. The sign of each term is positive or negative according as the number of factors I in front of the $A\{\}$ is even or odd. Certain theorems are obvious consequences but need not be detailed.

15. Definition. Let $Q(\alpha\beta)$ be any expression linear and homogeneous in the coordinates of α and β.

Also let

$$Q(\zeta\zeta) = \Sigma Q(e_i e_i) \qquad (i = 1 \ldots\ldots r)$$

be formed. This is called the Q-th bilinear ζ.[1]

16. Theorem. If e_i' is any other orthogonal system,

$$\begin{aligned} Q\,.\,\zeta\zeta &= \Sigma Q\,.\,e_j' e_k'\,.\,Ie_j' e_i\;\; Ie_k' e_i \\ &= \Sigma Q\,.\,e_j' e_k'\,.\,Ie_j' e_k' \\ &= \Sigma Q\,.\,e_i' e_i' \end{aligned}$$

Hence $Q\,.\,\zeta\zeta$ is independent of the orthogonal system.

It follows at once that

$$\rho = \zeta I\zeta\rho \qquad I\,.\,\zeta\zeta = r \qquad A\,.\,\zeta A\rho\zeta = (r-1)\rho$$
$$I\,.\,\zeta A\lambda_1\,A\zeta\mu_1 = (r-1)\,I\lambda_1\mu_1 \qquad A\,.\,\zeta\lambda_1\,A\zeta\mu_1\mu_2 = -(r-2)\,A\lambda_1\,A\mu_1\mu_2$$
$$I\,.\,\zeta A\lambda_1 \ldots\ldots \lambda_s\,A\zeta\mu_1 \ldots\ldots \mu_s = (r-s)\,I\,.\,\lambda_1\,A\lambda_2 \ldots\ldots \lambda_s\,A\mu_1 \ldots\ldots \mu_s$$
$$A\,.\,\zeta\lambda_1 \ldots\ldots \lambda_s\,A\zeta\mu_1 \ldots\ldots \mu_{s+1} = -(r-s-1)\,A\lambda_1 \ldots\ldots \lambda_s\,A\mu_1 \ldots\ldots \mu_{s+1}$$

$Q\,.\,\zeta\zeta$ may also be written $Q\,.\,\nabla\zeta$ by extending the definition of ∇, the coordinates of ζ being $x_1 \ldots\ldots x_r$, that is, $\nabla = \sum_{i=1}^{r} e_i \dfrac{\partial}{\partial x_i}$.

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[1] M'Aulay 1.

17. Theorem. By putting subscripts on the zeta-pairs we may use several. Thus

$$\rho = \zeta \,.\, I\zeta\rho = \zeta_1\, I\zeta_1\, \zeta_2\, I\zeta_2\, \rho$$
$$I\,.\,\zeta_1\,\zeta_2\, I\zeta_1\,\zeta_2 = r$$
$$I\,.\,\zeta_1\, A\zeta_2\, A\zeta_1\,\zeta_2 = r(r-1)$$
$$A\,.\,\zeta_1\,\zeta_2\, A\rho\zeta_1\,\zeta_2 = (r-2)\,(r-1)\,\rho$$
$$I\,.\,\zeta_1\, A\zeta_2\,\lambda_1\, A\zeta_1\,\zeta_2\,\mu_1 = (r-2)\,(r-1)\, I\lambda_1\,\mu_1$$
$$A\zeta_1\,\zeta_2\,\lambda_1\, A\zeta_1\,\zeta_2\,\mu_1\,\mu_2 = (r-3)\,(r-2)\, A\lambda_1\, A\mu_1\,\mu_2$$

In general

$$I\zeta_1\, A\zeta_2 \ldots \zeta_s\, A\zeta_1 \ldots \zeta_s = r\,(r-1) \ldots (r-s+1)$$

$$I\zeta_1\, A\zeta_2 \ldots \zeta_t\,\lambda_1 \ldots \lambda_s\, A\zeta_1 \ldots \zeta_t\,\mu_1 \ldots \mu_s =$$
$$(r-s)\,(r-s-1) \ldots (r-s-t+1)\; I\,.\,\lambda_1\, A\lambda_2 \ldots \lambda_s\, A\mu_1 \ldots \mu_s$$

If $s + t > r$ this vanishes; if $s + t = r$, we have

$$I\,.\,\lambda_1\, A\lambda_2 \ldots \lambda_s\, A\mu_1 \ldots \mu_s =$$
$$\frac{1}{(r-s)!}\; I\,.\,\zeta_1\, A\zeta_2 \ldots \zeta_t\,\lambda_1 \ldots \lambda_s\, A\zeta_1\,\zeta_2 \ldots \zeta_t\,\mu_1 \ldots \mu_s$$

$$A\zeta_1 \ldots \zeta_t\,\lambda_1 \ldots \lambda_{s-1}\, A\zeta_1 \ldots \zeta_t\,\mu_1 \ldots \mu_s =$$
$$(-1)^t\,(r-s) \ldots (r-s-t+1)\; A\lambda_1 \ldots \lambda_{s-1}\, A\mu_1 \ldots \mu_s$$

$$A\lambda_1 \ldots \lambda_{s-1}\, A\mu_1 \ldots \mu_s =$$
$$\frac{(-1)^{r-s}}{(r-s)!}\, A\zeta_1 \ldots \zeta_t\,\lambda_1 \ldots \lambda_{s-1}\, A\zeta_1 \ldots \zeta_t\,\mu_1 \ldots \mu_s\,.\; (s+t=r)$$

18. Theorem. If $I\alpha_i\,\rho = 0 \qquad i = 1 \ldots . \; m-1,$ then

$$\rho = A\,.\,\alpha_1 \ldots . \alpha_{m-1}\, A\beta_1 \ldots . \beta_m$$

where β_j, $(j = 1 \ldots . \; m)$ is arbitrary. For, if we take the case where $m - 1 = 3$, we have for β_1, β_2, β_3 all arbitrary, the identity

$$A\alpha_1\,\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\beta_3\,\rho = \beta_1\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_2\,\beta_3\,\rho - \beta_2\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_3\,\rho$$
$$+ \beta_3\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\rho - \rho I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\beta_3$$

Hence

$$\rho\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\beta_3 = \beta_1\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_2\,\beta_3\,\rho - \beta_2\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_3\,\rho$$
$$+ \beta_3\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\rho - A\alpha_1\,\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\beta_3\,\rho$$

Since $I\alpha_1\rho = 0 \qquad I\alpha_2\rho = 0 \qquad I\alpha_3\rho = 0$; therefore identically

$$I\alpha_1\beta_1\, I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_2\,\beta_3\,\rho - I\alpha_1\beta_2\, I\alpha_1\, A\alpha_2\alpha_3\, A\beta_1\,\beta_3\,\rho + I\alpha_1\beta_3\, I\alpha_1\, A\alpha_2\alpha_3\, A\beta_1\,\beta_2\rho = 0$$

with two similar equations for α_2, α_3. Therefore, since β_1, β_2, β_3 are arbitrary

$$I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_3\rho = 0 \qquad I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\rho = 0 \qquad I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_2\,\beta_3\,\rho = 0$$

or else, for any β_1, β_2, β_3,

$$I\alpha_1\, A\alpha_2\,\alpha_3\, A\beta_1\,\beta_2\,\beta_3 = 0$$

This is impossible, hence

$$\rho\, I\alpha_1 A\alpha_2\alpha_3 A\beta_1\beta_2\beta_3 = -A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3\rho$$
$$= -A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3(\rho + x\beta_1 + \ldots) = -A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3\beta_4$$

or

$$\rho = A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3\beta_4$$

where β_1, β_2, β_3, β_4 are arbitrary. A similar proof holds for the general case. This calculus would enable us to produce a theory of all bilinear functions $Q(\alpha\rho)$, and thus the so-called algebras.[1]

19. Definition. A subregion[2] consists of all hypercomplex numbers which can be expressed in the form $\alpha = a_1\rho_1 + a_2\rho_2 + \ldots + a_i\rho_i$ wherein ρ_1, ρ_2, $\ldots$, ρ_i are given, linearly independent, numbers of the range of the algebra.

20. Theorem. An unlimited number of groups of m independent numbers can be found in a region of m dimensions.[3] Any group is said to define the region.

21. Definition. The calculus of these entities is called an *algebra*, if it contains, besides addition, another kind of combination of its elements, called multiplication. The algebra is said to be of finite dimensions, when it depends on r units, r being a finite number. Of late the term finite has been applied to algebras the range of whose coordinates consists of a finite number of elements.

Multiplication is usually indicated by writing the numbers side by side, thus, $\alpha\beta$ or $\alpha.\beta$. Upon the definition of multiplication depends the whole character of the algebra.[4] The definition usually given is contained in the statements:

if $\quad \alpha = \sum_{i=1}^{r} a_i e_i \qquad \beta = \sum_{j=1}^{r} b_j e_j \qquad \gamma = \sum_{k=1}^{r} c_k e_k \qquad$ then $\alpha.\beta = \gamma$

if $\quad c_k = \sum_{i,j}^{1\ldots r} .\, a_i \,.\, b_j \,.\, \gamma_{ijk} \qquad (k = 1, 2 \ldots r)$

The constants γ_{ijk} are called *constants of multiplication*. If multiplication is defined in this manner the algebra is called *linear*. The products $a_i . b_j$ are defined for, and belong to, the range of coordinates. The constants of multiplication also belong to the range, and their products into $a_i b_j$ are defined for, and belong to the range. Algebras whose constants are such that $\gamma'_{jik} = \gamma_{ijk}$ are called *reciprocal*. If $\gamma'_{ikj} = \gamma_{ijk}$, they are *parastrophic*.

22. Theorem. If multiplication is defined as in § 21, then

$$\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma \qquad (\alpha + \beta) . \gamma = \alpha . \gamma + \beta . \gamma$$
$$(\alpha + \beta) . (\gamma + \delta) = \alpha . \gamma + \alpha . \delta + \beta . \gamma + \beta . \delta$$

This is usually called the *distributive* law of multiplication and addition. An algebra may be linear without being distributive.[5]

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[1] Whitehead 1, p. 123. [2] Cf. Whitehead 1, p. 123. [3] Cf. Gibbs 2, Macfarlane 4, Shaw 1.
[4] Shaw 9. [5] Dickson 7.

23. Definitions. In the product $\alpha\beta$, α is called the *facient*,[1] or the *left* factor, or the *prefactor*;[2] β is called the *faciend*,[1] or *right* factor, or *postfactor*.[2] The latter names will be used in this memoir.

If there is a number α_0 in the algebra, such that for every number of the algebra, α, $\alpha\alpha_0 = \alpha = \alpha_0\alpha$, then α_0 is called the *modulus*[3] of the algebra.

If we have $\quad \alpha\alpha_0 = \alpha \quad \alpha_0\alpha \neq \alpha \quad$ we may call α_0 a *post-modulus*.

If we have $\quad \alpha_0\alpha = \alpha \quad \alpha\,.\,\alpha_0 \neq \alpha \quad$ we may call α_0 a *pre-modulus*.

In defining an algebra, the existence of a modulus may or may not be assumed. When for all numbers α, β, we have $\alpha\beta = \beta\alpha$, the algebra is called *commutative*.

When for any three numbers α, β, γ we have $\alpha\,.\,(\beta\,.\,\gamma) = (\alpha\,.\,\beta)\,.\,\gamma$, the algebra is called associative.[4]

24. Theorem. If an algebra is linear, the product of any two numbers is known when the products of all the units are known. These products constitute the *multiplication table of the algebra.*

25. Theorem. In an associative algebra the constants of multiplication satisfy the law

$$\sum_{s=1}^{r} \gamma_{iks}\,\gamma_{sjt} = \sum_{s=1}^{r} \gamma_{kjs}\,\gamma_{ist} \qquad (i, k, j, t = 1 \,\ldots\, r)$$

26. Definitions. If $\alpha\,.\,\alpha \equiv \alpha^2 = \alpha$, then α is called *idempotent*.

If $\alpha^m = 0$, m a positive integer, then α is called *nilpotent*, of order[5] $m-1$. If $\alpha\beta = 0$, then α is *pre-nilfactorial* to β, which is *post-nilfactorial* to α.

If $\alpha\beta = 0 = \beta\alpha$, then α is *nilfactorial* to β, and β to α.

27. Definition. The expression $I.\,\alpha\beta$ is sometimes called the *inner* or *direct product*[6] of α, β and written $\alpha \cdot \beta$. Further, the expression

$$Q\,(\alpha\beta) \equiv \sum_{i,j}^{1\ldots r} a_i\,b_j\,.\,e_i\,I\,e_j\,()$$

is called the *dyadic* of $\alpha\beta$, and written $\alpha\beta$. It is thus an operator and not a product at all. The use of the term *product* in similar senses is quite common in the vector-analysis, but it would seem that it ought to be restricted to products which are of the same nature as the factors. GIBBS, however, insisted that any combination which was distributive over the coordinates of the factors was a product.[7]

There is no real difference between the theories of *dyadics*, *matrices*, *linear vector operators*, *bilinear forms*, and *linear homogeneous substitutions*, so far as the abstract theory is concerned and without regard to the operand.[8] If we

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[1] HAMILTON 1, B. PEIRCE 3. [2] TABER 5.

[3] SCHEFFERS 1, STUDY 1, who calls it *one* (Eins), identifying it with scalar unity. Some call it Haupt-einheit. Cf. SHAW 1.

[4] B. PEIRCE 3. [5] B. PEIRCE 3. [6] GIBBS 3. [7] GIBBS 2.

[8] FROBENIUS 1, and any bibliography of matrices, bilinear forms, or linear homogeneous substitutions. Cf. LAURENT 1, 2, 3, 4. See Chap. XXX this memoir.

denote the operator $Q(\alpha\beta)$ by ϕ, then the bilinear form $\Sigma c_{ij} x_i y_j$ may be written $I.\rho\phi\sigma$ or $I.\sigma\breve{\phi}\rho$, where $\breve{\phi}$ (or ϕ') is called the *conjugate*, the *transverse*, or the *transpose* of ϕ. Besides the ordinary combination of these operators by "multiplication" STEPHANOS[1] defines two other modes of composition which may be indicated as follows in the notation developed above:

(1) *Bialternate composition* in which

$\phi_1 \cdot \phi_2$ is equivalent to $\frac{1}{2!} C_{12}\, I\rho' A\rho''\, A\phi_1\sigma'\,\phi_2\sigma''$

$\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_s$ is equivalent to $\frac{1}{s!} C_{1\ldots s}\, I\rho' A\rho'' \ldots \rho^{(s)} A\phi_1\sigma' \ldots \phi_s\sigma^{(s)}$

$C_{1\ldots s}$ indicates that the sum is to be taken over all terms produced by permuting in every way the subscripts on the ϕ's.

(2) *Conjunction*, which corresponds to the multiplication of algebras, and is equivalent to taking ϕ_1 and ϕ_2 on different independent grounds $e_1 \ldots e_r$, $e_1' \ldots e_{r'}'$, whose products $e_i e_j'$ define a new ground

$$e_{ij} = e_i e_j' \qquad (i = 1 \ldots r,\ j = 1 \ldots r')$$

Thus
$$\phi_1 \times \phi_2 = \sum_{i,j}^{1\ldots r} \sum_{k,l}^{1\ldots r'} c_{ij}^{(1)} c_{kl}^{(2)} e_{ik}\, I e_{jl}$$

2. DEFINITIONS BY INDEPENDENT POSTULATES.

28. Definition. Three definitions by postulates proved to be independent have been given by DICKSON.[2] The latest definition is as follows:

A set of n ordered marks $a_1 \ldots a_r$ of F (a field) will be called an *n-tuple element* a. The symbol $a = (a_1 \ldots a_r)$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1 \ldots a_r = b_r$.

A system of n-tuple elements a in connection with n^3 fixed marks γ_{ijk} of F will be called a *closed system* if the following five postulates hold.

Postulate I: If a and b are any two elements of the system, then $s = (a_1 + b_1 \ldots a_r + b_r)$ is an element of the system.

Definition: Addition of elements is defined by $a \mathbin{\dot{+}} b = s$.

Postulate II: The element $0 = (0 \ldots 0)$ occurs in the system.

Postulate III: If 0 occurs, then to any element a of the system corresponds an element a' of the system, such that $a \mathbin{\dot{+}} a' = 0$.

Theorem: The system is a commutative group under $\dot{+}$.

Postulate IV: If a and b are any two elements of the system, then $p = (p_1 \ldots p_r)$ is an element of the system, where

$$p_i = \sum_{jk}^{1\ldots r} a_j b_k \gamma_{jki} \qquad (i = 1 \ldots r)$$

Definition: Multiplication of elements is defined by $a \otimes b = p$.

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[1] STEPHANOS 6. [2] DICKSON 5, 8.

Postulate V: The fixed marks γ satisfy the relations

$$\sum_{j=1}^{r} \gamma_{stj}\, \gamma_{jki} = \sum_{j=1}^{r} \gamma_{tkj}\, \gamma_{sji} \qquad (s, t, k, i = 1 \ldots\ldots r)$$

Theorem: Multiplication is associative and distributive.

Postulate VI: If $\tau_1 \ldots\ldots \tau_r$ are marks of F such that $\tau_1 a_1 + \ldots + \tau_r a_r = 0$ for every element $(a_1 \ldots\ldots a_r)$ of the system, then $\tau_1 = 0 \ldots\ldots \tau_r = 0$. [This postulate makes the system r *dimensional*].

Theorem: The system contains r elements $\varepsilon_i = (a_{i1} \ldots\ldots a_{ir})$, $i = 1 \ldots\ldots r$ such that $|a_{ij}| \neq 0$.

Theorem: Every r-dimensional system is a complex number system.

Generalization: If the marks $a_1 \ldots\ldots a_{r_1}$ belong to a field F_1; and if $a_{r_1+1} \ldots\ldots a_{r_1+r_2}$ belong to a field F_2; $\ldots\ldots$, if a corresponding change is made in postulate VI; if further $\gamma_{jki} = 0$, when j, k, i, belong to different sets of subscripts, then we have a closed system not belonging to a field F.[1]

3. DEFINITIONS IN TERMS OF LOGICAL CONSTANTS.

29. This definition is recent, and due to BERTRAND RUSSELL. By logical constants is meant such terms as *class, relation, transitive relation, asymmetric relation, whole and part,* etc. Complex numbers are defined in connection with *dimensions,* or the study of *geometry.* The definition in its successive parts runs as follows:[2]

30. Definition. By *real* number is meant any integer, rational fraction, or irrational number, defined by a sequence. These have been discussed previously, in the work referred to.

A hypercomplex number is an aggregate of r one-many relations, the series of real numbers being correlated with the first r integers. Thus, to the r integers we correlate $a_1, a_2 \ldots\ldots a_r$, all in the range of real numbers. This correlation is expressed by the form

$$a_1 e_1 + a_2 e_2 + \ldots\ldots + a_r e_r$$

The order of writing the terms may or may not be essential to the definition. The e indicates the correlation, thus e_1 is not a unit, but a mere symbol, the unit being $1e_1$. The remaining definitions, addition, multiplication, etc. may be easily introduced on this basis.

Theorem: Hypercomplex numbers may be arranged in an r-dimensional series.

31. A like logical definition may be given when the elements belong to any other range than that of "real" numbers.

4. ALGEBRAIC DEFINITION.

32. The preceding definitions are of entities essentially multiplex in character. The units either directly or implicitly are in evidence from the

[1] Cf. CARSTENS 1.

[2] B. RUSSELL 1, pp. 378–379.

beginning. It seems desirable to avoid this multiplicity idea, or implication, until the development itself forces it upon us. Historically this is what happened in Quaternions. Originally quaternions were operators and their expressibility in terms of any independent four of their number was a matter of deduction, while HAMILTON always resisted the *coordinate* view. The following may be called the algebraic definition, since it follows the lines of certain algebraic developments.

33. Definition. Let there be an assemblage of entities A_i, either finite or transfinite, enumerable or non-enumerable. They are however well-defined, that is, distinguishable from one another. Further, let these entities be subject to processes of deduction or inference, such that from two entities, A_i, A_j, we deduce by one of these processes, passing from A_i to A_j, the entity A_k; which we will indicate by the expression

$$A_i\, OA_j = A_k \qquad (A_i,\, A_j \text{ any elements of the assemblage})$$

A different process O' would generally lead to a different entity A'_k; thus

$$A_i\, O'A_j = A'_k$$

(These processes may be, for example, addition $\oplus$, and multiplication $\otimes$). It is assumed that these processes and their combinations are fully defined by whatever postulates are necessary. Then the entities A_i and the processes O, O' .... are said to form a *calculus*, and the assemblage of entities will be called a *range*.

34. Definition. Let there be given a range and its calculus, and let us suppose the totality of expressions of the calculus are at hand. In certain of these, M_1, $M_2 \ldots M_r$, let us suppose the constituent entities A_i, A_j .... are held as fixed, and that we reduce the totality of expressions modulo these expressions M; that is, wherever these expressions occur in any other expression, they are cancelled out. Then the calculus so taken modulo M is called an *algebra*.

For example, let the range A be all rational numbers. Let the expressions M be

$$M\begin{Bmatrix} i + 1 \\ j^2 + 1 \end{Bmatrix}$$

Then an expression like $4 - 8$ may be written $4i + 4 + 4 - 8 = 4i$; an expression like $x^2 + 9$ becomes $x^2 + 9 - (9 + 9j^2) = x^2 - 9j^2$; which may be factored into $(x + 3j)\,(x - 3j)$ or $(x + 3j)\,(x + 3ij)$.

In this manner we have a calculus in which will always appear the elements i, j (or j and j^2 as we might find by reductions). Modulo $i + 1$ and $j^2 + 1$, certain expressions become reducible, that is factorable, which otherwise cannot be factored. We call the expressions xi, xj, xj^3, in this case, where x is any rational number, *negative numbers*, *imaginary numbers*, and *negative imaginary numbers*. We consider i and j as *qualitative units*, although perhaps *modular units* would be a better term.

35. It is not assumed necessarily that there is but one entity A_i for any given expression, for we may have two expressions alike except as to the elements that enter them. Thus we might have

$$M \left\{ \begin{matrix} i^2 + 1 \\ j^2 + 1 \end{matrix} \right\}$$

36. Definition. In any case we shall call the expressions M the *defining expressions* of the algebra, and the elements A_i (such as i, j) entering them the *fundamental qualitative units.*

37. Postulates :

I. It is assumed that the processes of the calculus are *associative.*

II. It is assumed that the processes which shall furnish the defining expressions shall be those called *addition* $\oplus$, and *multiplication* $\otimes$.

III. It is assumed that the process $\otimes$, multiplication, is *distributive* as to the process $\oplus$, addition. That is

$$A_i \otimes (A_j \oplus A_k) = (A_i \otimes A_j) \oplus (A_i \otimes A_k)$$
$$(A_j \oplus A_k) \otimes A_i = (A_j \otimes A_i) \oplus (A_k \otimes A_i)$$

38. The commutativity of multiplication is not assumed. Further, the general question of processes and their relations is discussed, so far as it bears on these topics, in XIII, hence will not be detailed here.

It is evident according to this definition that an algebra may spring from an algebra. Hence the term is a relative one, and indeed we may call a calculus an algebra if we consider that the calculus is really taken modulo

$$A_i \, O \, A_j - A_k \qquad A_i \, O' A_j - A'_k, \text{ etc.}$$

That is, the equalities or substitutions allowed in the calculus make it an algebra. The only calculus in fact there is, is the calculus of all entities A_i, A_j, A_k, etc., which permits no combinations, that is, no processes, at all. From $A_i, A_j, \ldots$ we infer or derive nothing at all, not even zero. The calculus of symbolic logic is thus properly an *algebra.*

Any definition of an algebra must reduce to this definition ultimately, for the multiplication-table itself is a set of r^2 defining expressions. That is, we work modulo[1]

$$e_i e_j - \sum_{k=1}^{r} \gamma_{ijk} e_k \qquad (i, j = 1 \ldots . r)$$

39. Definition. If the range of an algebra can be separated into r sub-ranges, each of which is a sub-group under the process of addition $\oplus$; so that an entity which is the sum of elements from each of the sub-ranges is not reducible to any entity which is a sum of elements from some only of the sub-ranges; then the algebra is said to be (additively) r-dimensional.

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[1] Cf. Kronecker 1, where this view is very clearly the basis for commutative systems.

40. It is to be noted that an algebra may be r-dimensional and yet have in it $r+s$ distinct qualitative units. Thus, ordinary positive and negative numbers form an algebra of two units but of only one dimension. Ordinary complex numbers contain four qualitative units, but form an algebra of two dimensions.

The defining expressions determine the question of dimensionality. For example, let the defining expressions be

$$\begin{cases} e_1^3-1 \qquad e_2^2-1 \qquad e_2 e_1 - e_1^2 e_2 \\ e_1+e_1^2+1 \end{cases}$$

whence we may add

$$e_1 e_2 + e_1^2 e_2 + e_2 \qquad e_2 e_1 e_2 e_1 - 1 \qquad e_1 e_2 e_1 e_2 - 1 \qquad e_1 e_2 e_1^2 e_2 e_1 - 1$$

We have here two more defining expressions than are needed to define an algebra of six units, hence the algebra becomes four-dimensional. The problem of how many defining expressions are necessary to define an algebra of r units has never been generally solved even for such simple algebras as abstract groups. If the algebra is finite of order r, a maximum value for the number is r^2. But a single expression may define an infinite algebra.

Nothing, so far as known to the writer, has been done towards the study of these algebras of deficient dimensionality.

II. THE CHARACTERISTIC EQUATION OF A NUMBER.

41. Theorem. Any number ζ in a finite linear associative algebra which contains a modulus, e_0, and whose coordinates range over all scalars, satisfies identically an equation of the form $\Delta'(\zeta)=0$, and equally an equation of the form $\Delta''(\zeta)=0$. In each case, $\Delta'(\zeta)$ or $\Delta''(\zeta)$ is a polynomial in ζ of order r, the order of the algebra.[1]

The function $\Delta'.\zeta$, called the *pre-latent function*[2] of ζ, has the form

$$\Delta'.\zeta = \begin{vmatrix} \Sigma.x_i\gamma_{i11}e_0-\zeta & \Sigma.x_i\gamma_{i21} & \dots & \Sigma.x_i\gamma_{ir1} \\ \Sigma.x_i\gamma_{i12} & \Sigma.x_i\gamma_{i22}e_0-\zeta & \dots & \Sigma.x_i\gamma_{ir2} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma.x_i\gamma_{i1r} & \Sigma.x_i\gamma_{i2r} & \dots & \Sigma.x_i\gamma_{irr}e_0-\zeta \end{vmatrix}$$

The function $\Delta''.\zeta$, called the *post-latent function*[2] of ζ, has the form

$$\Delta''.\zeta = \begin{vmatrix} \Sigma.x_i\gamma_{1i1}e_0-\zeta & \Sigma.x_i\gamma_{2i1} & \dots & \Sigma.x_i\gamma_{ri1} \\ \Sigma.x_i\gamma_{1i2} & \Sigma.x_i\gamma_{2i2}e_0-\zeta & \dots & \Sigma.x_i\gamma_{ri2} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma.x_i\gamma_{1ir} & \Sigma.x_i\gamma_{2ir} & \dots & \Sigma.x_i\gamma_{rir}e_0-\zeta \end{vmatrix}$$

[1] The relation between this equation and the corresponding equation for matrices is so close that we may include in one set references to both: CAYLEY 3; LAGUERRE 1; B. PEIRCE 1, 3; FROBENIUS 1, 2; SYLVESTER 1, 2, 3; BUCHHEIM 3; SCHEFFERS 1, 2, 3; WEYR 1, 5, 8; TABER 1, 4; PASCH 1; MOLIEN 1; CARTAN 2; SHAW 4.

[2] Cf. TABER 1.

In each case Σ stands for $\sum\limits_{i=1}^{r}$. These functions may be expanded according to powers of ζ, taking the forms

$$\Delta'.\zeta = \zeta^r - m_1'\,\zeta^{r-1} + m_2'\,\zeta^{r-2}\ldots\ldots\ldots + (-)^r\, m_r'\, e_0$$
$$\Delta''.\zeta = \zeta^r - m_1''\,\zeta^{r-1} + m_2''\,\zeta^{r-2}\ldots\ldots\ldots + (-)^r\, m_r''\, e_0$$

In certain cases (viz., when the algebra is equivalent to its reciprocal) these two become identical. (The absence of a modulus does not add to the generality of the treatment.) These equations exist for all ranges of coordinates.

42. **Definition.** The coefficients m_1' and m_1'' are respectively the *pre-scalar* and the *post-scalar*[1] of ζ, multiplied by r; that is, if we designate the scalar of ζ by $S.\zeta$, we have

$$S'.\zeta = \frac{m_1'}{r} \qquad S''.\zeta = \frac{m_1''}{r}$$

If we indicate $S.\zeta^i$ by S_i, we have by well-known relations from the theory of algebraic equations

$$i\,!\,m_i = \begin{vmatrix} r\,S_1 & 1 & 0 & 0 & \ldots & 0 \\ r\,S_2 & r\,S_1 & 2 & 0 & \ldots & 0 \\ r\,S_3 & r\,S_2 & r\,S_1 & 3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r\,S_{i-1} & r\,S_{i-2} & r\,S_{i-3} & & \ldots & i-1 \\ r\,S_i & r\,S_{i-1} & r\,S_{i-2} & & \ldots & r\,S_1 \end{vmatrix}$$

Theorem: The symbol S obeys the laws[2]

$$S(\zeta + \sigma) = S\zeta + S\sigma \qquad S.a\zeta = a\,S\zeta \qquad (a, \text{ any scalar}) \quad S.e_0 = 1$$

43. **Definition.** The number $V'.\zeta \equiv \zeta - S'.\zeta$ is the *pre-vector*[3] of ζ, and the number $V''.\zeta \equiv \zeta - S''.\zeta$ is the *post-vector*[3] of ζ. By substituting these for ζ in the identity for m_i in § 42 we arrive at various interesting and useful formulae.

44. **Definition.** If the two equations $\Delta'.\zeta = 0$, $\Delta''.\zeta = 0$ are not identical, the process of finding the highest common factor will lead to a new expression $\Delta.\zeta$ which must vanish. When the two equations $\Delta'.\zeta = 0$, $\Delta''.\zeta = 0$ are identical we may also have ζ satisfying an equation of lower order; let the lowest such be

$$\Delta.\zeta = 0$$

This single equation is called the *characteristic equation* of ζ, and $\Delta.\zeta$ is the *characteristic function* of ζ.[4] [The pre-latent equation was called *the identical equation* by CAYLEY, *characteristic* by FROBENIUS and MOLIEN, and this lower

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[1] TABER 2, 3, 4, 5. Cf. FROBENIUS 14, § 4. FROBENIUS called m_1 the *Spur* of ζ.
[2] TABER 2, 3, 4, 5.
[3] Cf. TABER 2, 3.
[4] See references to § 41.

equation has been called "*Ranggleichung*" by MOLIEN, "*Grundgleichung*" by WEYR, *identical equation* and *fundamental equation* by TABER, *characteristic equation* by SCHEFFERS, and in some cases it is the *reduced characteristic* equation.]

45. Theorem. The characteristic function is a factor of the two latent functions.[1]

46. Definition. The order of the characteristic function being $r' \leqq r$, it may be written

$$(\zeta - g_1 e_0)^{\mu_1} \ldots\ldots (\zeta - g_p e_0)^{\mu_p} \qquad \mu_1 + \ldots\ldots + \mu_p = r'$$

The scalars $g_1 \ldots\ldots g_p$ are the p distinct *latent roots* of ζ. The exponents $\mu_1 \ldots\ldots \mu_p$ are the p *sub-multiplicities* of the roots of ζ. The factor $\zeta - g_i e_0$ is the *latent factor*[2] of the root g_i.

WEIERSTRASS called $(\zeta - g_1 e_0)$, and powers, *elementary factors* (*elementartheiler*), particularly the powers: $k_{\mu_i i}\, k_{\mu_i - 1, i} \ldots\ldots k_{1, i}$. See MUTH 1 for references to this subject, or WEIERSTRASS 1; KRONECKER 2, 3, 4; FROBENIUS, *Crelle* 86, 88; *Berliner Sitz-ber.* 1890, 1894, 1896.

47. Theorem. For a fixed integer i $(1 \leqq i \leqq p)$, there is at least one solution, σ, $(\sigma \neq 0)$ for each of the equations

$$\begin{array}{l} (\zeta - g_i e_0)\ \sigma = 0 \\ (\zeta - g_i e_0)^2 \sigma = 0 \\ \ldots\ldots\ldots\ldots \\ (\zeta - g_i e_0)^{\mu_i} \sigma = 0 \end{array}$$

A solution of the k-th equation is a solution of those that follow. If σ_{ikt} is a solution of the k-th of these equations, then among the solutions of the $k+1$-th equation, which include the solutions of the previous equations, some are linearly independent of the entire set of solutions σ_{ikt} of the k-th equation.[3]

Theorem: The solutions of these equations for different values of i are linearly independent of each other.[4]

48. Definition. The number

$$Z_i \equiv \frac{(\zeta - g_1 e_0)^{\mu_1} \ldots (\zeta - g_{i-1} e_0)^{\mu_{i-1}} (\zeta - g_{i+1} e_0)^{\mu_{i+1}} \ldots (\zeta - g_p e_0)^{\mu_p}}{(g_i - g_1)^{\mu_1} \ \ldots\ (g_i - g_{i-1})^{\mu_{i-1}} \ \ (g_i - g_{i+1})^{\mu_{i+1}} \ \ldots (g_i - g_p)^{\mu_p}}$$

is the i-th *latent* of ζ; it corresponds to the root g_i. There are thus p latents of ζ.

49. Theorems. The product of Z_i and any number of the algebra is either zero or else it is a number in the region of solutions of the equations in § 47.[5] We may symbolize this by writing $Z_i\{\sigma\} = \{\zeta_i\}$ The region $\{\zeta_i\}$ is called the i-th pre-latent region of ζ. There are correspondingly post-latent regions of ζ.

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[1] TABER 1; WEYR 8; MOLIEN 1; FROBENIUS 14. [2] TABER 1; WHITEHEAD 1.
[3] TABER 1; WHITEHEAD 1; CARTAN 2. [4] TABER 1; WHITEHEAD 1; SHAW 4. [5] SHAW 4.

The p latent regions of ζ together constitute the whole domain of the algebra.[1] It is obvious that the Z's are such that if $i \neq j$,

$$Z_i Z_j = 0 \qquad (Z_i - e_0)^{\mu_i} Z_i = 0$$

50. **Theorem.**[2] The p pre-(post-) latent regions are linearly independent, that is, mutually exclusive, and together define the ground of the algebra. Each latent factor annuls its own latent region but does not annul any part of any other latent region. The i-th pre-latent region may not contain the same numbers as the i-th post-latent region. The dimensions of the i-th pre-latent region are given by the exponent of the i-th latent factor as it appears in the pre-latent equation. The pre-latent equation contains as factors only the latent factors to multiplicities μ_i', such that

$$\mu_i' \geqq \mu_i \qquad (i = 1 \ldots . p)$$
$$\sum_{i=1}^{p} \mu_i' = r$$

Likewise the post-latent equation contains as factors only the latent factors to multiplicities μ_i'', such that

$$\mu_i'' \geqq \mu_i \qquad (i = 1 \ldots . p)$$
$$\sum_{i=1}^{p} \mu_i'' = r$$

51. **Theorem.**[3] The pre- (post)- latent region $\{\zeta_i\}$ contains μ_i sub-latent regions $\{\Sigma_{i1}\}$, $\{\Sigma_{i2}\}$, $\ldots$, $\{\Sigma_{i\mu_i}\}$, where each sub-latent region includes those of lower order, say $\{\Sigma_{ik}\}$ includes $\{\Sigma_{ik'}\}$ if $k' < k$.

The region $\{\Sigma_{ik}\}$ is such that $(\zeta - g_i e_0)^k \{\Sigma_{ik}\} = 0$, but in $\{\Sigma_{ik}\}$ is at least one number σ_{ik} for which $(\zeta - g_i e_0)^{k-1} \sigma_{ik} \neq 0$.

52. **Definition.** For brevity let $\zeta - g_i e_0 \equiv \theta_i$; then, in $\{\zeta_i\}$, $\theta_i^{\mu_i}$ annuls certain independent numbers which no lower power of θ_i annuls. Let these be w_{1i} in number, represented by

$$\xi_{11}^{1i} \quad \xi_{21}^{1i} \cdots\cdots \xi_{w_{1i}1}^{1i}$$

Of course any w_{1i} independent numbers linearly expressible in terms of these would answer as well to define this region, so that only the region is unique. Then each of these multiplied by θ_i gives a new set of w_{1i} numbers independent of each other and of the first set. Let these be

$$\theta_i \,.\, \xi_{s1}^{1i} \equiv \xi_{s2}^{1i} \qquad (s = 1 \ldots . w_{1i})$$

In general we shall have for the products by powers of θ_i a set of numbers linearly independent of each other,

$$\theta_i^h \,.\, \xi_{s1}^{1i} \equiv \xi_{s\,h+1}^{1i} \qquad \begin{cases} h = 0 \ldots . \mu_i - 1 \\ s = 1 \ldots . w_{1i} \end{cases}$$

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[1] Taber 1; Whitehead 1; Shaw 4.

[2] Taber 1; Whitehead 1; Shaw 4; Weyr 8; Buchheim 3, 7, 9.

[3] See preceding references.

The region made up of, or defined by, these numbers will be called the *first pre-shear region*[1] of the i-th latent region. It may be represented by $\{X'_{1i}\}$. Let there be chosen now out of the numbers remaining in the i-th latent region, w_{2i} linearly independent numbers which are annulled by that next lower power of θ_i, say μ_{i2}, which annuls these w_{2i} numbers, but such that $\theta_i^{\mu_{i2}-1}$ does not annul them and such that $\theta_i^{\mu_{i2}+1}$ does not annul any number which $\theta_i^{\mu_{i2}}$ does not also annul. These numbers and their products by powers of θ_i give rise to the *second pre-shear region*, $(\mu_{i2} < \mu_i)$

$$\{X'_{2i}\} \equiv \{\theta_i^h \xi_{s1}^{2i}\} \qquad \begin{cases} h = 0 \dots \mu_{i2} - 1 \\ s = 1 \dots w_{2i} \end{cases}$$

We proceed thus, separating the i-th latent region into c_i shear regions, $\{X'_{1i}\}, \dots, \{X'_{c_i i}\}$, containing respectively $(\mu_i = \mu_{i1})$ $\mu_{i1} w_{1i}, \dots, \mu_{ic_i} w_{c_i i}$ linearly independent numbers, with

$$\mu'_i = \sum_{j=1}^{c_i} . \mu_{ij} w_{ji}$$

There is a corresponding definition for the post-regions.

53. Theorem.[2] The pre- and the post-latent equations are (using accents as before to distinguish the two sets of numbers)

$$\prod_{i=1}^{p} \theta_i^{\Sigma w'_{ji} \mu'_{ij}} = 0 \qquad j = 1 \dots c'_i$$

$$\prod_{i=1}^{p} \theta_i^{\Sigma w''_{ji} \mu''_{ij}} = 0 \qquad j = 1 \dots c''_i$$

54. Theorem. If all the roots g_i vanish, ζ is a nilpotent, and for some power μ we have $\zeta^\mu = 0$.

Further, for every number there are exponents μ'_j, μ''_j, such that

$$\zeta^{\mu'_j} \sigma = 0 = \sigma \zeta^{\mu''_j} \qquad \mu'_j \leqq \mu, \ \mu''_j \leqq \mu$$

If ζ and σ are of the same character,[3] $(\alpha\alpha)$ then for any power μ_k, $\zeta^{\mu'_k} \sigma$ and $\sigma \zeta^{\mu_k}$ are nilpotent.

The product may not be nilpotent if ζ is of character $(\alpha\beta)$ and σ of character $(\beta\alpha)$. If the product is not nilpotent the algebra contains at least one quadrate. If an algebra contains no quadrates, $\zeta^{\mu_k} \sigma$ and $\sigma \zeta^{\mu_k}$ are nilpotent for all values[4] of σ and μ_k.

55. Definitions. When the coefficients in the pre-latent (post-latent) equation vanish in part so that

$$m_j = 0 \qquad j > r - \mu'_0$$

then ζ is said to have *vacuity*[5] of order μ'_0. There are μ'_0 zero-roots, and one or more solutions of the equations

$$\zeta \sigma = 0 \qquad \zeta^2 \sigma = 0 \qquad \dots\dots \qquad \zeta^{\mu'_0} \sigma = 0 \qquad \mu_0 \leqq \mu'_0$$

[1] SHAW 4 [2] SHAW 4. [3] See § 59. [4] CARTAN 2; TABER 4. [5] SYLVESTER 1; TABER 1.

The solutions of $\zeta\sigma = 0$ define the *null-region* of ζ. The number of independent numbers in this region (its dimensions) is the *first nullity* of ζ, say k_1. The k_2 independent solutions of $\zeta^2\sigma = 0$, $\zeta\sigma \neq 0$, define the first sub-null-region of ζ, of second nullity k_2; proceeding thus we have[1]

$$k_1 \geqq k_2 \geqq \dots \geqq k_{\mu_0}$$

The vacuity of course is given by the equation

$$\mu'_0 = k_1 + \dots + k_{\mu_0}$$

The characteristic equation, it must be remembered, contains ζ^{μ_0} as a factor; the pre-latent equation $\zeta^{\mu'_0}$, the post-latent $\zeta^{\mu''_0}$. The partitions of μ'_0 which satisfy the inequalities above give all the possible ways in which the sub-null-regions can occur.

56. **Theorem.**[2] Each latent factor, ζ_i, is a number whose pre-latent (post-latent) equation will contain $\zeta_i^{\mu'_i}$, and whose characteristic equation will contain $\zeta_i^{\mu_i}$. The nullities of ζ_i are given by the equations

$$\begin{aligned}
k_{\mu_i i} &= w_{1i} \\
k_{\mu_i - 1 i} &= w_{1i} + t w_{2i} & t &= 0 \text{ or } 1 \\
&\dots\dots\dots \\
k_{2i} &= w_{1i} + w_{2i} + \dots + t_1 w_{c_i - 1 i} + t_2 w_{c_i i} & t_1,\ t_2 &= 0 \text{ or } 1 \\
k_{1i} &= w_{1i} + w_{2i} + \dots + w_{c_i i}
\end{aligned}$$

The vacuity $\mu'_i = \mu_{i1} w_{1i} + \mu_{i2} w_{2i} + \dots + \mu_{i c_i} w_{c_i i}$

57. **Theorem.** The number ζ may be written[3]

$$\zeta = \sum_{i=1}^{p} \{ g_i x_i + h^{i1} \phi_i + h_{i2} \phi_i^2 + \dots + h_{i\,\mu_{i1}-1} \phi_i^{\mu_{i1}-1} \}$$

wherein the numbers $x_i \quad \phi_i \quad (i = 1 \dots p)$ satisfy the following laws:

$$\begin{aligned}
x_i^2 &= x_i & x_i x_j &= 0 & \text{if } i &\neq j \\
\phi_i^{\mu_{i1}} &= 0 & \phi_i \phi_j &= 0 & \text{if } i &\neq j \\
x_i \phi_j &= 0 = \phi_j x_i & & & \text{if } i &\neq j \\
x_i \phi_i &= \phi_i = \phi_i x_i
\end{aligned}$$

The numbers x_i and ϕ_i are all linearly independent and belong to the algebra, at least if we have coordinates ranging over the general scalar field.

58. **Theorem.**[4] Let $h_{i1} \phi_i + \dots + h_{i\,\mu_{i1}-1} \phi_i^{\mu_{i1}-1} \equiv \Phi_i$; then if $F(x)$ is any analytic function of x, $F'(x) \dots$ its derivatives,

$$F.\zeta = \sum_{i=1}^{p} \left\{ F(g_i) \cdot x_i + F'(g_i) \cdot \Phi_i + \frac{F''(g_i)}{2!} \Phi_i^2 + \dots \frac{F^{\mu_{i1}-1}(g_i)}{(\mu_{i1}-1)!} \Phi^{\mu_{i1}-1} \right\}$$

[1] Sylvester 2; Taber 1; Buchheim 9; Whitehead 1.
[2] § 52.
[3] Study 6; Shaw 7.
[4] Shaw 7. Cf. Taber 1; Sylvester 3.

59. Theorem. The different numbers of the algebra will yield a set of idempotent expressions $e_1 \ldots\ldots e_a$, such that if $i \neq j$, $i, j = 1 \ldots\ldots a$

$$e_i^2 = e_i \qquad e_i e_j = 0 = e_j e_i \qquad e_0 = e_1 + \ldots\ldots + e_a$$

and hence the numbers of the algebra may be divided into classes $\{Z_{\alpha\beta}\}$, such that if $\zeta_{\alpha\beta}$ is in the class $\{Z_{\alpha\beta}\}$, then

$$e_s \zeta_{\alpha\beta} e_t = \vartheta_{s\alpha} \vartheta_{\beta t} \zeta_{\alpha\beta}$$

The subscripts α, β are the *characters*[1] (pre- and post- resp.) of $\zeta_{\alpha\beta}$. In this and similar expressions $\vartheta_{xy} = 0$ when $x \neq y$, $\vartheta_{xy} = 1$ when $x = y$.

60. Theorem. The product of $\zeta_{\alpha\beta}$ and $\zeta_{\gamma\delta}$ is given (when it does not vanish on account of properties not dependent on the characters) by the equation[2]

$$\zeta_{\alpha\beta} \zeta_{\gamma\delta} = \vartheta_{\beta\gamma} \zeta_{\alpha\delta}$$

The numbers $\zeta_{\alpha\alpha}$ form a sub-algebra, $(\alpha = 1, \ldots, a)$.

61. Theorem. Let the characteristic equation of ζ have $q-1$ distinct roots which are not zero, and let $\nu-1$ be the lowest power of ζ in this equation. Then if

$$F.\zeta \equiv e_0 - \prod_{i=1}^{q-1} \left(e_0 - \frac{\zeta^\nu}{g_i^\nu}\right)^{\mu_i}$$

we have[3]

$$F.\zeta = \sum_{i=1}^{q-1} x_i$$

62. Theorem.[4] If $e_0 \neq \sum_{i=1}^{q-1} x_i$, then $e_0 = \sum_{i=1}^{q-1} x_i + x_q$, where x_q belongs to the root zero and

$$x_q = \prod_{i=1}^{q-1} \left(e_0 - \frac{\zeta^\nu}{g_i^\nu}\right)^{\mu_i}$$

Theorem: It also follows, that, if

$$F_i \zeta = \prod_{\substack{j=1 \\ j \neq i}}^{p} e_0 \left[1 - \left(\frac{\zeta - g_i}{g_j - g_i}\right)^{\mu_i}\right]^{\mu_j}$$

then $$F_i \zeta = x_i$$

63. The use of the two sets of idempotents of ζ, the pre- and the post-, enables us to find partial moduli, which are not necessarily invariant, and the modulus, which is invariant.

For example, let us have the algebra

	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_2	e_3
e_3	e_3	0	0

Then $$e_0 = e_1 + e_2$$

[1] SCHEFFERS 1, 2, 3; CARTAN 2; HAWKES 1; SHAW 4. Cf. B. PEIRCE 1, 3. FROBENIUS 14.
[2] See references to § 59. [3] TABER 4. [4] TABER 4.

If we put $\zeta = e_1 + e_3$, we find

$$\zeta(e_1 + e_3) = (e_1 + e_3) \qquad \zeta \cdot e_2 = 0 \qquad \zeta \cdot e_3 = 0$$

hence the characteristic equation $(\zeta - e_0)\,\zeta = 0$, and by §§ 61, 62,

$$x_1 = \zeta \qquad x_2 = e_2 - e_3 \qquad e_0 = x_1 + x_2$$

These determine the same algebra (in the sense of invariant equivalence)

	x_1	x_2	e_3
x_1	x_1	0	0
x_2	0	x_2	e_3
e_3	e_3	0	0

and the partial moduli are not the same as before, being e_1, e_2 in one case, $e_1 + e_3$, $e_2 - e_3$ in the other.

64. **Theorem.** If ζ_i is any number in the i-th pre- (post-) region of ζ, and if σ is any number of the algebra, then $\zeta_i\,\sigma$ $(\sigma\,\zeta_i)$ is a number wholly in the i-th pre- (post-) region.[1] Consequently the numbers in the i-th pre- (post-) region form a sub-algebra.

65. **Theorem.** Let the numbers defining the i-th post-latent region of ζ be ξ^{ji}_{st}, where

$$j = 1 \;\ldots\; c_i \qquad s = 1 \;\ldots\; w_{ji} \qquad t = 1 \;\ldots\; \mu_{ij}$$

We have of course

$$\xi^{ji}_{st} \cdot \theta_i = \xi^{ji}_{st+1}$$

so that

$$\xi^{ji}_{st} = \xi^{ji}_{s1} \cdot \theta_i^{t-1}$$

Then by § 64 the product of any number σ gives

$$\sigma \cdot \xi^{ji}_{s1} = \Sigma \cdot a^{ki}_{uv}\, \xi^{ki}_{uv}$$

Hence when these coefficients a are known we know the product of σ into any number of the form ξ^{ji}_{st}, for

$$\sigma \cdot \xi^{ji}_{st} = \Sigma \cdot a^{ki}_{uv}\, \Sigma^{ki}_{uv+t-1}$$

where[2] ξ^{ki}_{uv+t-1} must be zero if $v + t - 1 > \mu_{ik}$.

66. **Theorem.** If τ is any number of the algebra which satisfies the equation $\tau \cdot \theta_i^x = 0$, where $\tau \cdot \theta_i^{x-1} \neq 0$; then τ must be in the region (§ 13) Σ''_{is}, and in no lower region.[3]

67. **Theorem.** If τ is any number of the algebra, and if σ_{is} lies in the region Σ''_{is} but in no lower region, then $\tau\sigma_{is}$ lies at most in the region Σ''_{is}, and may lie wholly in lower regions.[4]

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[1] Shaw 4. [2] Shaw 4. [3] Shaw 4. [4] Shaw 4.

68. Theorem. Let Σ^{i1} be the region to which $.\ \theta^{(\mu_{i1}-1)}$ reduces the whole i-th post-latent region, and generally Σ^{is} be the region to which $.\ \theta^{(\mu_{i1}-s)}$ reduces the latent region. Then if τ is any number of the algebra, and σ^{is} any number of the region Σ^{is}, then

$$\tau\sigma^{is} = {}'\sigma^{is}, \text{ a number of the region } \Sigma^{is} \text{ or lower regions.}^{1}$$

69. Theorem. If σ^{is}_{it} is a number common to both regions Σ^{is} and Σ_{it}, then $\tau\ .\ \sigma^{is}_{it} = {}'\sigma^{is}_{it}$, a number in the same regions.[2]

70. Theorem. Let

$$S^1_{\mu_{ij}} \text{ be the region } \{\xi^{ji}_{s1}\}, \qquad (s = 1 \ldots . w_{ji})$$
$$S^a_b \equiv S^a_{\mu_{ij}-a+1} = S^1_{\mu_{ij}}\ .\ \theta_i^{a-1}$$

then $S^a_b \equiv S^a_{\mu_{ij}-a+1}$ belongs to the regions $\Sigma^{i,\mu_{i1}-a+1}$ and $\Sigma_{i,\mu_{ij}-a+1}$. Then if τ is any number, $\tau\ .\ S^a_b = \{S^{a'}_{b'} \ldots . S^{a^{(t)}}_{b^{(t)}} \ldots .\}$ for all values of t subject to the conditions

$$a^{(t)} \geqq a \qquad b^{(t)} \leqq b \qquad a^{(t)} + b^{(t)} = \mu_{ij} + 1$$

This may also be expressed in the following statement:

$$\tau\ .\ \xi^{ji}_{st} = \Sigma a^{ki}_{xy}\ \xi^{ki}_{xy}$$

where $y \leqq \mu_{ik}$, and ξ^{ki}_{xy} belongs to $S^y_{\mu_{ik}-y+1}$, and ξ^{ji}_{st} belongs to $S^t_{\mu_{ij}-t+1}$.

Hence

$$y \geqq t \qquad \mu_{ik} - y \leqq \mu_{ij} - t$$

that is

$$y \geqq t + \mu_{ik} - \mu_{ij}$$

Or finally,[3] if $\mu_{ik} \leqq \mu_{ij}$, then $\mu_{ik} \geqq y \geqq t$

if $\qquad \mu_{ik} > \mu_{ij}$, then $\mu_{ik} \geqq y \geqq t + \mu_{ik} - \mu_{ij}$

It is to be remembered also that

$$a^{ki}_{xy} = a^{ki}_{xy-1} = \ldots .. = a^{ki}_{xy-t+1}$$

It is evident that the products into ξ^{ji}_{s1} determine all the other products.

71. Theorem. Since the units of the algebra may be the numbers ξ^{ji}_{st}, as these are mutually independent and r in number, it follows that among the n^3 constants of the algebra, γ, which the coefficients a reduce to in this case, there are many which vanish and many which are equal. The units may be so chosen in any algebra that the corresponding constants γ become subject to the equations for the coefficients a in § 70 [but this choice may introduce irrational transformations].

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[1] SHAW 4. [2] SHAW 4. [3] SHAW 4.

72. Theorem. Since the idempotents for ζ, viz., $x_1, x_2, \ldots x_p$, may be used as pre-multipliers as well as post-multipliers, the units ξ^{ji}_{s1}, and therefore all units, may be separated into parts according to the products

$$x_\alpha \cdot \xi^{ji}_{s1} \qquad (\alpha = 1 \ldots p)$$

As these parts are linearly independent, and as the i-th region is defined already by the units ξ^{ji}_{s1}, it follows that the independent units derived by this pre-multiplication must also define the region, and as the shear regions were unique, their number for each shear remains the same as before. We may use a new notation, then, indicating the pre- as well as the post-character of ξ, and at the same time uniting j and s into a single subscript, thus the units are

$$\theta^{t'-1}_{i'} \left({}^{i'}_{u'\,t'}\, \xi^{i''}_{u''\,t''}\right) \theta^{t''-1}_{i''}$$

where

$$u'' = w_{1i''} + \ldots + w_{j''i''} + s \qquad u' = w_{1i'} + \ldots + w_{j''i''} + s$$

73. Theorem. Let us return to the equation in § 70, in the new notation,

$$\tau \cdot {}^{\alpha}\xi^{\beta}_{ut} = \Sigma\, a^{(\iota\beta)}_{xy}\, {}^{\iota}\xi^{\beta}_{xy}$$

$$\mu_{\beta x} \geqq y \geqq t \qquad y \geqq t + \mu_{\beta x} - \mu_{\beta u} \qquad a^{\iota\beta}_{xy} = a^{\iota\beta}_{xy-1} = \ldots a^{\iota\beta}_{xy-t+1}$$

If τ is confined to expressions belonging to the region $\{\xi^{\alpha\alpha}_{u1}\}$, then letting $\tau^{\alpha\alpha}_{1}$ be any such number,

$$\tau^{\alpha\alpha}_{1} \cdot {}^{\alpha}\xi^{\alpha}_{u1} = \Sigma\, a^{\alpha\alpha}_{x1}\, {}^{\alpha}\xi^{\alpha}_{x1} + \Sigma\, a^{\alpha\alpha}_{x1}\, {}^{\alpha}\xi^{\alpha}_{xy}$$

$$\mu_{\alpha x} > y > 1 \qquad y \geqq 1 + \mu_{\alpha x} - \mu_{\alpha u}$$

If we let

$$\sigma^{\alpha\alpha}_{1} = \Sigma\, z^{\alpha\alpha}_{u}\, {}^{\alpha}\xi^{\alpha}_{u1} \qquad u = 1, \ldots, w_{1\alpha} + w_{2\alpha} + \ldots + w_{c_\alpha\alpha}$$

then

$$\tau^{\alpha\alpha}_{1} \cdot \sigma^{\alpha\alpha}_{1} = \Sigma\, a^{\alpha\alpha}_{x1}\, z^{\alpha\alpha}_{u}\, {}^{\alpha}\xi^{\alpha}_{x1} + \text{terms for which } y > 1$$

Hence if we let $\tau^{\alpha\alpha}_{1}$ be in turn each unit ${}^{\alpha}\xi^{\alpha}$ in this region, we shall find from $\sigma^{\alpha\alpha}_{1}$ by the process used in the beginning of the problem, certain numbers idempotent so far as this region is concerned, and which will be linearly expressible in terms of ${}^{\alpha}\xi^{\alpha}_{u1}$. These new x's are linearly independent and commutable with x_α, since, if x'_α is one of them, $x_\alpha x'_\alpha = x'_\alpha = x'_\alpha x_\alpha$. Hence x_α must be the sum of them. We might therefore have chosen for ζ a number which would have had these idempotents, and we may suppose that the number ζ has been so chosen that no farther subdivision of the idempotents is possible.[1]

74. Theorem. It is evident that, as the expressions in the i-th latent region of ξ form a sub-algebra, we may choose one of them ξ'_i just as we choose ξ,

[1] Cf. Molien 1.

and using it as a post-multiplier, divide this i-th latent region itself into sub-regions corresponding to the latent regions of ξ_i' in $\{\xi_i\}$. Each such sub-region becomes a sub-algebra. We may evidently so proceed subdividing the whole algebra into sub-regions until ultimately no sub-region contains any number which used as pre-multiplier has more than one root for that sub-region. This root may then be taken as zero or unity. If then the sub-region be represented by $\sigma_1, \sigma_2 \ldots \sigma_{r'}$, we have for every number

$$\sigma = \Sigma x_j \sigma_j \qquad \tau = \Sigma y_j \sigma_j \qquad (j = 1 \ldots r')$$
$$\tau\sigma = g\tau + \tau' \qquad \tau'\sigma = g\tau' + \tau''$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\tau^{(h-1)}\sigma = g\tau^{(h-1)} + \tau^{(h)} \qquad \tau^{(h)}\sigma = g\tau^{(h)}$$

Hence if $\varkappa_i$ is the partial modulus for this region defined by

$$\varkappa_i = \frac{\sigma^h - hg\sigma^{h-1} + \dfrac{h(h-1)}{1\,.\,2}\,g^2\sigma^{h-2} \ldots + (-)^{h-1}g^{h-1}\sigma}{g^h}$$

we must have $\sigma = g\varkappa_i + \vartheta_i +$ other terms whose post-product by τ is zero.

Multiplying every number then by $\varkappa_i\,.\,()$ we arrive at a sub-sub-region which gives a sub-algebra whose modulus is $\varkappa_i$, and such that if α is its character, every number in it has the character

$$(\alpha\,\alpha)$$

This algebra is a Peirce algebra. Its structure will be studied later. The Peirce algebra is the ultimate subdivision by this method of the algebra in general and its structure really determines the main features of the structure of the general algebra.

75. An algebra may contain an infinity of units, in which case it may not have an equation at all. Thus the algebra may have for units

$$e_0 \qquad e_1 \qquad e_1^2 \ldots\ldots e_1^r \qquad e_1^{r+1} \ldots\ldots$$

so that
$$\rho = \sum_{i=0}^{\infty} x_i e_i$$

It may very well happen then that $\rho\sigma = t\sigma$ has no solution. The theory of such algebras will be developed in a later paper.

76. Theorem. Let the general equation of a number ζ be

$$\zeta^r - m_1\zeta^{r-1} + \ldots \qquad + (-1)^r m_r e_0 = 0$$

Let us put $\zeta^2 - m_1\,\zeta + \sigma = 0$. Then we may eliminate ζ from these two

equations, by using determinants, arriving at an equation in terms of σ of order r. Thus we have

$$\begin{vmatrix} 1 & -m_1 & +m_2 & -m_3 \ldots\ldots & (-1)^r m_r & 0 \\ 0 & 1 & -m_1 & m_2 \ldots\ldots & (-1)^{r-1} m_{r-1} & (-)^r m_r \\ 1 & -m_1 & \sigma & 0 \ldots\ldots & 0 & 0 \\ 0 & 1 & -m_1 & \sigma \ldots\ldots & 0 & 0 \\ 0 & 0 & 1 & -m_1 \ldots\ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \ldots\ldots & \sigma & 0 \\ 0 & 0 & 0 & 0 \ldots\ldots & -m_1 & \sigma \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 1 & -m_1 & +m_2 & \ldots \\ 0 & 1 & -m_1 & \ldots \\ 0 & 0 & \sigma - m_2 & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 = \begin{vmatrix} \sigma - m_2 & m_3 & -m_4 & \ldots \\ 0 & \sigma - m_2 & m_3 & \ldots \\ 1 & -m_1 & \sigma & \ldots \\ 0 & 1 & -m_1 & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

The highest powers of this equation are

$$\sigma^r - 2m_2 \sigma^{r-1} + \ldots = 0$$

Hence the sum of the roots of σ is $2m_2$.

77. **Theorem.** In general if $\zeta^s - m_1 \zeta^{s-1} + \ldots + (-1)^s \sigma_s = 0$, we find in the same manner a determinant of order $r + s$, reducing to one of order r in σ_s, the first two terms becoming

$$\sigma_s^r - s m_s \sigma_s^{r-1} + \ldots = 0$$

Hence for any such number

$$(-1)^{s-1} \sigma_s = (\zeta^s - m_1 \zeta^{s-1} + \ldots + (-)^{s-1} m_{s-1} \zeta) = \zeta \chi^{(s)}(\zeta)$$

we have the sum of the roots of σ_s equal to sm_s. Hence the *s-th scalar coefficient* m_s of ζ is $\frac{1}{s}$ into the scalar coefficient of order unity of $(-)^{s-1} \zeta \chi^{(s)}(\zeta)$; or

$$m_s = \tfrac{1}{s}(-1)^{s-1} m_1 [\zeta \chi^{(s)}(\zeta)]$$

78. **Theorem.** We may also find the general equations of the σ's, and in a similar way of the χ's.

79. **Theorem.** In this way one may form the equations of powers of ζ, or of any polynomial in ζ.

80. **Theorem.** Let there be formed for any number ρ, the products

$$\rho e_i \qquad (i = 1 \ldots r)$$

The e_i form the basis and are orthogonal. Then we have $(\rho - g)\sigma = 0$, when

$$\sum_{j=1}^{r} . I . e_i (\rho e_j) I . e_j \sigma = g I . e_i \sigma \qquad (i = 1 \ldots r)$$

Hence

$$\begin{vmatrix} I.e_1(\rho e_1)-g & I.e_1(\rho e_2)\ldots\ldots \\ I.e_2(\rho e_1) & I.e_2(\rho e_2)-g\ \ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}=0$$

or

$$g^r - g^{r-1}\sum_{i=1}^{r}.I.e_i(\rho e_i)+g^{r-2}\sum_{i,j=1}^{r} I.e_i\,Ae_j\,A(\rho e_i)\,(\rho e_j)-\ldots.=0$$

This, however, must correspond to the general pre-latent equation of ρ, and therefore

$$m_1'=\sum_{i=1}^{r}.I.e_i(\rho e_i) \qquad m_2'=\sum_{i,j=1}^{r}.I.e_i\,Ae_j\,A(\rho e_i)\,(\rho e_j)\ \ldots\ldots \text{etc.}$$

81. Theorem. We have at once

$$\chi'.\sigma=(m_1'-\rho).\sigma=\sum_{i=1}^{r}.(\sigma I.e_i(\rho e_i)-(\rho e_i)\,I.e_i\sigma)$$
$$=\sum_{i=1}^{r}.A.e_i\,A\sigma(\rho e_i)$$

Thus

$$\rho\chi'.\sigma=\sum_{i,j}(\rho e_j)\,I.e_j\,Ae_i\,A\sigma(\rho e_i)$$

Therefore

$$2!\ m_2'=\sum_k.I.e_k(\rho.\chi'e_k)=\Sigma I.e_k(\rho e_j)\,I.e_j\,Ae_i\,Ae_k\,(\rho e_i)$$
$$=\Sigma.I.(\rho e_j)\,e_k\,I.e_k\,A(\rho e_i)\,Ae_j\,e_k$$
$$=\Sigma.I.(\rho e_j)\ A(\rho e_i)\,Ae_j\,e_i$$
$$=\sum_{i,j}.I.e_i\,Ae_j\,A(\rho e_i)\,(\rho e_j)$$

Since $\chi''.\sigma=(m_2'-\rho.\chi')\sigma$, we have

$$\chi''.\sigma=\sum_{i,j}A.e_i\,e_j\,A\sigma(\rho e_i)\,(\rho e_j)$$

In general, we find

$$\chi^{(s)}.\sigma=\Sigma.Ae_i\,e_j\ldots.e_s\,A\sigma(\rho e_i)\,(\rho e_j)\ \ldots.\ (\rho e_s)$$

82. Theorem. If we use the notation of the ζ-pairs, these become

$$\rho\sigma=(\rho\zeta)\,I.\zeta\sigma$$
$$m_1'=I\zeta(\rho\zeta)$$
$$m_s'=\frac{1}{s!}I.\zeta_1\,A\zeta_2\ldots.\zeta_s\,A(\rho\zeta_1)\,(\rho\zeta_2)\ldots.(\rho\zeta_s)$$
$$\chi^{(s)}.\sigma=\frac{1}{s!}A\zeta_1\ldots.\zeta_s\,A\sigma(\rho\zeta_1)\ldots.(\rho\zeta_s)$$

In this form, the independence of the expressions m and χ from any particular unit-system is shown.

83. Theorem. Let us write further

$$m'(\rho_1,\ \rho_2\ldots.\rho_s)=\frac{1}{s!}I\zeta_1\,A\zeta_2\ldots.\zeta_s A(\rho_1\zeta_1)\,(\rho_2\zeta_2)\ldots.(\rho_s\zeta_s)$$

Then, from the properties of the ζ's, this form will reduce to

$$m'(\rho_1,\ldots.,\rho_s)=\frac{1}{s!}\big[m_1'(\rho_1)\,m_1'(\rho_2)\ldots.m_1'(\rho_s)$$
$$-\Sigma.m_1'(\rho_1)\,(m_1'(\rho_2)\ldots.m_1'(\rho_{s-1}\rho_s)+\ldots.\big]$$

according to the rule: Insert m_1' before every selection of ρ's taken according to the partitions of s, giving each term of $s - h$ factors the sign $(-)^h$, and writing in each factor the product of the ρ's in every order possible when the ρ of lowest subscript is kept first in the product. For example, if $s = 3$, we have the partitions $3 = 1 + 1 + 1 = 1 + 2 = 3$. Hence,

$$m'(\rho_1, \rho_2, \rho_3) = \frac{1}{3!}[m_1(\rho_1)\, m_1'(\rho_2)\, m_1'(\rho_3) - m_1'(\rho_1)\, m_1'(\rho_2\rho_3) - m_1'(\rho_2)\, m_1'(\rho_1\rho_3) - m_1'(\rho_3)\, m_1'(\rho_1\rho_2) + m_1'(\rho_1\rho_2\rho_3 + \rho_1\rho_3\rho_2)]$$

We note that

$$m'(\rho_a\rho_b \cdots \rho_e\rho_f) = m'(\rho_f\rho_a\rho_b \cdots \rho_e)$$

If $s = r + 1$, this form must vanish identically.

84. Theorem. If we put

$$\chi(\rho_1 \cdots \rho_s)\,\sigma = \frac{1}{s!}A \,.\, \zeta_1 \cdots \zeta_s A\,\sigma(\rho_1\zeta_1)(\rho_2\zeta_2)\cdots(\rho_s\zeta_s)$$

then, if χ_1 stands for $[\chi(\rho_1)](\sigma)$, χ_{12} for $[\chi(\rho_1\rho_2)]\,\sigma$, etc.,

$$\chi(\rho_1 \cdots \rho_s)\,\sigma = \frac{1}{s!}[\chi_1 \cdot \chi_2 \cdots \chi_s(\sigma) - \Sigma \cdot \chi_1 \cdot \chi_2 \cdots \chi_{s-1,s}(\sigma) + \cdots]$$

The rule is the same as for the preceding expression of m', thus

$$\chi(\rho_1, \rho_2, \rho_3) \cdot \sigma = \frac{1}{3!}[\chi_1 \cdot \chi_2 \cdot \chi_3(\sigma) - \chi_1\chi_{23}(\sigma) - \chi_2\chi_{13}(\sigma) - \chi_3\chi_{12}(\sigma) + (\chi_{123} + \chi_{132})\,\sigma]$$

85. Theorem. If $m_{s_1, s_2, \ldots, s_t}$ is the function $\Sigma \cdot g_1^{s_1} g_2^{s_2} \cdots g_t^{s_t}$, summation over all permutations of $1, 2 \ldots t$, then

$$m_{s_1, s_2, \ldots, s_t} = I\zeta_1 A\,\zeta_2 \cdots \zeta_t A\,(\rho^{s_1}\zeta_1)(\rho^{s_2}\zeta_2)\cdots(\rho^{s_t}\zeta_t)$$

86. These numbers m and functions χ are called *invariants* of ρ, or of $\rho_1, \rho_2 \cdots$, as the case may be, since they do not depend on any particular system of units. It is obvious that any function of $\rho_1, \rho_2 \cdots \rho_t$, containing only ζ-pairs, is an *invariant*[1] in this sense.

87. Theorem. If $\rho\alpha = 0$, then $\chi' \cdot \alpha = m_1\alpha$, $\qquad \rho\chi' \cdot \alpha = m_1\rho\alpha = 0$

$$\chi'' \cdot \alpha = m_2\alpha$$
$$\chi^{(s)} \cdot \alpha = m_s \cdot \alpha$$

In general, if $\rho\alpha = g\alpha$, then

$$\chi \cdot \alpha = (m_1 - g)\,\alpha \cdots\cdots \chi^{(s)} \cdot \alpha = (m_s - m_{s-1}g + \cdots \pm g^s)\,\alpha$$

If $\qquad \rho\alpha_1 = g\alpha_1 + \alpha_2 \qquad \rho\alpha_2 = g\alpha_2$

$$\chi^{(s)} \cdot \alpha_1 = (m_s - m_{s-1}g + \cdots \pm g^s)\,\alpha_1 - (m_{s-1} - m_{s-2}g \cdots \mp g^{s-1})\,\alpha_2$$

Similar results may be found for the other latent regions of ρ.

[1] Cf. M'Aulay 1.

III. THE CHARACTERISTIC EQUATIONS OF THE ALGEBRA.

88. Theorem. Of the units taken to define the algebra in the preceding chapter, certain ones will be of pre-character α, post-character β. Let the number of such be represented by $n_{\alpha\beta}$. Then the total number of those of post-character β will be

$$n''_\beta \equiv n_{1\beta} + n_{2\beta} + \dots + n_{p\beta}$$

The number of pre-character α will be

$$n'_\alpha = n_{\alpha 1} + n_{\alpha 2} + \dots + n_{\alpha p}$$

89. Theorem. We may state the general multiplication theorem again in the following form, ζ being any number:

$$\zeta \cdot {}^{\alpha}\zeta^{\beta}_{jk} = \sum_{i=1}^{p} \cdot a^{i\beta}_{j'jk'-k} \, {}^{i}\zeta^{\beta}_{j'k'}$$

where

$$k' - k \geqq 0 \qquad \mu_{ij'} - k > k' - k \geqq \mu_{ij'} - \mu_{\alpha j}$$

In this equation each coefficient a is a linear homogeneous function of certain of the coordinates x of ζ, namely those of type $x^{(i\alpha)}_{uv}$ where ${}^{i}\xi^{\alpha}_{uv}$ combines with ${}^{\alpha}\xi^{\beta}_{jk}$ without vanishing.

90. Theorem. If we multiply ζ into each unit, and form the equations resulting from the pre-latent equation[1] of ζ, say $\Delta' \cdot \zeta = 0$, we have at once, because the units have been chosen for the post-regions of a certain number ξ,

$$\Delta' \cdot \zeta = \Delta'_1 \zeta \cdot \Delta'_2 \zeta \dots \Delta'_p \zeta$$

The orders of these determinant factors are n''_1, $n''_2 \dots n''_p$, their sum being equal to r.

91. Theorem. An examination of the determinant Δ'_i shows that it may be divided into blocks by horizontal and vertical lines, which separate the different units ${}^{i}\xi^{i}_{u1}$, ${}^{i}\xi^{i}_{u2}$, ... according to the power of θ_i which produces the units, the order being

$$\xi_{u1} \dots \xi_{u\mu_{i1}}$$

There are μ_{i1} columns and rows of blocks. But, from the properties of the coefficients a, the constituents in the first block on the diagonal are the only constituents in any block on the diagonal. Hence we may write[2]

$$\Delta'_i = \Delta'^{\mu_{i1}}_{i1} \Delta'^{\mu_{i2}}_{i2} \dots \Delta'^{\mu_{ic_i}}_{ic_i}$$

92. Theorem. The determinants Δ'_{is}, $s = 1 \dots c_i$, are irreducible in the coordinates of ζ, so long as ζ is *any* number. For, if one of these determinants were reducible, then the original separation by idempotents could have been pushed farther—as this separation was assumed to be ultimate no farther reduction is possible.[3]

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[1] On the general equation see STUDY 2, 3; SFORZA 1, 2; SCHEFFERS 1, 2, 3; MOLIEN 1; CARTAN 2; SHAW 4; TABER 4; FROBENIUS 14.

[2] SHAW 4. Cf. CARTAN 2.

[3] Cf. CARTAN 2.

93. Theorem. Confining the attention to Δ'_{ik}, let the units ${}^{a}\xi^{i}_{j1}$ whose products by ζ give Δ'_{ik}, be h in number, with the pre-characters $\alpha = 1, \ldots, f$, $f \leqq h$. The coordinates x appearing in the coefficients a, must be of the form $x^{(\alpha_1 \alpha_2)}$. It follows that if ζ be chosen so that all coordinates x not of these characters $(\alpha_1 \alpha_2)$, $\alpha_1, \alpha_2 = 1 \ldots f$, are zero, then the value of Δ'_{ik} will not be affected. The aggregate of such numbers, however, obviously constitute a subalgebra which includes $\varkappa_a$, $\alpha = 1 \ldots f$. These numbers, say $\zeta^{\alpha_1 \alpha_2}$, when multiplied together yield a pre-latent equation $\delta_{aa} = 0$, which must be a power of Δ'_{ik}, and therefore irreducible. It follows that if we treat this subalgebra as we have the general case, we shall find but one shear making up the whole of each latent region. Consequently the units of this algebra take the form

$$e_{\alpha_1 \alpha_2} \qquad (\alpha_1, \alpha_2 = 1 \ldots f)$$

They may be so chosen that

$$e_{\alpha_1 \alpha_2} \cdot e_{\alpha_3 \alpha_4} = \vartheta_{\alpha_2 \alpha_3} \cdot e_{\alpha_1 \alpha_4}$$

The partial moduli are evidently[1]

$$e_{\alpha_1 \alpha_1} \qquad (\alpha_1 = 1 \ldots f)$$

94. Theorem. Since any unit ${}^{a}\xi^{\beta}$ may be written $e_{aa}\, {}^{a}\xi^{\beta}$ it follows that no expression $e_{\alpha_1 a}\, {}^{a}\xi^{\beta}$ can vanish, else

$$e_{a\alpha_1}\, e_{\alpha_1 a}\, {}^{a}\xi^{\beta} = e_{aa}\, {}^{a}\xi^{\beta} = 0$$

Hence if there is one unit ${}^{a}\xi^{\beta}$, there are all the units[2]

$$e_{\alpha_1 a}\, {}^{a}\xi^{\beta} = {}^{\alpha_1}\xi^{\beta} \qquad (\alpha_1 = 1 \ldots f)$$

95. Theorem. The units of the algebra may therefore be represented by the symbols

$$e^{(j)}_{\alpha\beta}\, e_{\beta\gamma}\, e^{(k)}_{\gamma\delta}$$

where the numbers $e^{(j)}_{\alpha\beta}$ and $e^{(k)}_{\gamma\delta}$ are such that

$$e^{(j)}_{\alpha\beta}\, e^{(k)}_{\gamma\delta} = \vartheta_{jk}\, \vartheta_{\beta\gamma}\, e^{(j)}_{\alpha\delta}$$

The numbers $e_{\beta\gamma}$ form an algebra by themselves, such that its equation consists of linear factors only,[3] as

$$\Delta_{ij} = (a^{(ii)}_{jj0} - \zeta)$$

96. Definition. An algebra whose equation contains only linear factors will be called a Scheffers algebra. If, further, it contains but one linear factor, it will be called a Peirce algebra. If it contains factors of orders higher than unity, it will be called a Cartan algebra. An algebra consisting of units of the type $e^{(j)}_{\alpha\beta}$ only, will be called a Dedekind algebra.[4] The *degree* of an algebra is the order of its characteristic equation in ζ.

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[1] Molien 1 (ursprüngliche systeme); Cartan 1, 2; Shaw 4; Frobenius 14.

[2] Cartan 2; Frobenius 14.

[3] Cartan 1, 2. On the "multiplication" of algebras by each other, see Clifford 8; Taber 1; Scheffers 3. Cf. Taber 4; Hawkes 1, 2; Frobenius 14.

[4] On classification see Scheffers 3, 4; Molien 1, 2, 3; Cartan 1, 2; Shaw 4; B. Peirce, 1, 3.

97. Theorem. Let the algebra be of the Scheffer's type. The irreducible factors of its pre-latent equation are all linear; hence in the latent post-region of any root of ξ, the shears are of width unity only. The units defining the i-th region become

$$^{\alpha}\xi^{i}_{jt} \qquad \alpha = 1 \ldots . p \qquad j = 1 \ldots . c_i \qquad t = 1 \ldots . \mu_{ij}$$
$$\mu_{i1} > \mu_{i2} \ldots . > \mu_{ic_i}$$

The product of ζ into any unit is[1]

$$\zeta \cdot {}^{\alpha}\xi^{i}_{jt} = \Sigma\, a^{(\beta\alpha)}_{j'j\,k'-k}\; {}^{\beta}\xi^{i}_{j'k'}$$

where

$$k' - k \geqq 0 \qquad \mu_{\beta j'} - k > k' - k \geqq \mu_{\beta j'} - \mu_{ij} \qquad j' \leqq j \text{ if } k' = k$$

98. Theorem. If we remove from this algebra all idempotent units, the remaining units form a nilpotent algebra of $r - p$ dimensions. The equation $\Delta' \zeta = 0$ reduces in this case to a determinant whose constituents on the diagonal and to the right of the diagonal all vanish, hence it is evident that the product of any two of its numbers is expressible in terms of at most $r - p - 1$ numbers. Let the original units be $\phi_{p+1}, \phi_{p+2} \ldots . \phi_r$. Then the products $\phi_{t_1}\phi_{t_2}$ do not contain a certain region defined by a set of units

$$\phi_{p+1} \quad \ldots . \quad \phi_{p+h_1} \qquad\qquad (h_1 > 0)$$

The products of these h_1 units (which constitute the region ε_1, let us say,) among themselves and with any other units, are linearly expressible in terms of

$$\phi_{p+h_1+t} \qquad\qquad (t = 1 \ldots . r - p - h_1)$$

Similarly any product $\phi_{t_1}\ \phi_{t_2}\ \phi_{t_3}$ can not contain a region ε_2, defined by

$$\phi_{p+1} \ldots ., \phi_{p+h_1} \qquad \phi_{p+h_1+1} \ldots . \phi_{p+h_1+h_2} \qquad\qquad (h_2 > 0)$$

Hence $\{\varepsilon_1\}\cdot\{\varepsilon_2\}$, $\{\varepsilon_2\}\cdot\{\varepsilon_1\}$, and $\{\varepsilon_2\}\cdot\{\varepsilon_2\}$ depend only on ϕ_{p+t}, $t > h_1 + h_1$. Proceeding thus, it is evident the domain of the nilpotent algebra may be separated into regions defined by classes of units which give products of the form

$$\{\varepsilon_i\} \cdot \{\varepsilon_j\} = \{\varepsilon_k\} \qquad\qquad (k > i,\ k > j)$$

In particular, the units of the Scheffer's nilpotent algebra may always be chosen so that, if they are η_i, η_j $\ldots .$, then

$$\eta_i\,\eta_j = \Sigma\, \gamma_{ijk}\,\eta_k \qquad\qquad (k > i,\ k > j)$$

It is also evident that for any $r - p + 1$ numbers ζ_t we have

$$\zeta_1 \cdot \zeta_2 \ldots . \zeta_{r-p+1} = 0$$

The[2] products of order l form a sub-algebra of order r',

$$r' < r - l + 2$$

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[1] SHAW 4, 5.

[2] SCHEFFERS 3; CARTAN 2; SHAW 5; FROBENIUS 14.

99. Theorem. In any Cartan algebra the units may be so taken as to be represented by

$$e^{(i)}_{\alpha\beta} \qquad (i = 1 \ldots . p \qquad \alpha, \beta = 1 \ldots . w_i)$$
$$\eta^{(ij)}_{\alpha\beta k} \qquad (i, j = 1 \ldots . p \qquad \alpha, \beta = 1 \ldots . w_i)$$

The laws of multiplication[1] are

$$e^{(i)}_{\alpha\beta}\, e^{(j)}_{\gamma\delta} = \vartheta_{ij}\, \vartheta_{\beta\gamma}\, e^{(i)}_{\alpha\delta}$$
$$e^{(i)}_{\alpha\beta}\, \eta^{(i'j)}_{\gamma\delta k} = \vartheta_{\beta\gamma}\, \vartheta_{ii'}\, \eta^{(ij)}_{\alpha\delta k}; \quad \eta^{(ij)}_{\alpha\beta k}\, e^{(j')}_{\gamma\delta} = \vartheta_{jj'}\, \vartheta_{\beta\gamma}\, \eta^{(ij)}_{\alpha\delta k}$$
$$\eta^{(ij)}_{\alpha\beta k}\, \eta^{(i'j')}_{\alpha'\beta'k'} = \vartheta_{ji'}\, \vartheta_{\beta\alpha'}\, \eta^{(ij')}_{\alpha\beta'k''}; \qquad k'' > k,\ k'' > k'$$

100. Theorem. Returning to the Scheffers algebra, if we retain only its nilpotent sub-algebra and the modulus, we shall have a Peirce algebra. The equation of this algebra will contain but a single factor and the pre- and post-characters of its units may be assumed to be the same. The nilpotent θ becomes the sum of the nilpotents $\theta_1 + \theta_2 \ldots . + \theta_p$. The product of ζ into any unit may be written[2]

$$\zeta \cdot \xi_{jk} = \Sigma\, a_{j'jk'-k}\, \xi_{j'k'}$$
$$k' - k \geqq 0 \qquad \mu_{j'} - k > k' - k \geqq \mu_{j'} - \mu_j \qquad j' \leqq j \text{ if } k' = k$$

101. Theorem. Let the characteristic equation of any number be

$$\zeta^m - f_1 \cdot \zeta^{m-1} + \ldots . + (-)^m f_m = 0 \qquad (m \leqq r)$$

where f_i is a homogeneous function of the coordinates of order i. Differentiating this equation, and remembering that $d\zeta$ is *any* number, we arrive at m general equations connecting 1, 2, ..., m numbers of the algebra: as

$$(\zeta_1^{m-1}\zeta_2 + \zeta_1^{m-2}\zeta_2\zeta_1 + \ldots . \zeta_2\zeta_1^{m-1}) - [f_1(\zeta_2)\,\zeta_1^{m-1}$$
$$+ f_1(\zeta_1) \cdot \zeta_1^{m-2}\zeta_2 + \ldots .] + \ldots . = 0$$
$$(\zeta_1^{m-2}\zeta_2\zeta_3 + \ldots .) - \text{etc.} = 0$$

These equations are the *second*, *third*, etc. *derived* equations of the algebra, according as they contain two, three, etc., independent numbers ζ_1, ζ_2, etc. These equations lead to many others when the scalars of ζ are introduced.[3] The new coefficients $f_i(\zeta_{a_1} \ldots . \zeta_{a_i})$ will be called the scalar characteristic coefficients of order i for $\zeta_{a_1} \ldots . \zeta_{a_i}$. They usually differ from the coefficients m.

102. Theorem. The general equation of r numbers of the algebra of order r is written (Σ representing the sum of the r! terms got by permuting all the subscripts)

$$\Sigma(\zeta_1\zeta_2\zeta_3 \ldots \zeta_r) - \Sigma(m_1\zeta_1 \cdot \zeta_2\zeta_3 \ldots \zeta_r) + \Sigma(m_2(\zeta_1, \zeta_2) \cdot \zeta_3 \ldots \zeta_r) - \ldots$$
$$+ (-1)^r m_r \cdot \zeta_1\zeta_2 \ldots \zeta_r = 0$$

[1] Cartan 2. [2] Shaw 4, 5. [3] Taber, 2, 3. Shaw 4.

In this equation, omitting the subscript 1, so that $m \equiv m_1$

$$\begin{aligned}
m_2(\zeta_i, \zeta_j) &= m\zeta_i \cdot m\zeta_j - m\zeta_i\zeta_j = m_2(\zeta_j, \zeta_i) \\
m_3(\zeta_i, \zeta_j, \zeta_k) &= m\zeta_i \cdot m\zeta_j \cdot m\zeta_k - m\zeta_i \cdot m\zeta_j\zeta_k - m\zeta_j \cdot m\zeta_i\zeta_k \\
&\qquad - m\zeta_k \cdot m\zeta_i\zeta_j + m\zeta_i\zeta_j\zeta_k + m\zeta_i\zeta_k\zeta_j \\
&= m_3(\zeta_i, \zeta_k, \zeta_j) = m_3(\zeta_j, \zeta_k, \zeta_i) \\
&= m\zeta_i \cdot m_2(\zeta_j, \zeta_k) + m\zeta_j \cdot m_2(\zeta_i, \zeta_k) + m\zeta_k \cdot m_2(\zeta_i, \zeta_j) \\
&\qquad - 2m\zeta_i \cdot m\zeta_j \cdot m\zeta_k + m\zeta_i\zeta_j\zeta_k + m\zeta_i\zeta_k\zeta_j
\end{aligned}$$

These formulæ follow from the identities

$$\begin{aligned}
sm_s(\phi_1, \phi_1 \dots \phi_1) = m_1 [m_{s-1}(\phi_1, \phi_1 \dots \phi_1) \cdot \phi_1 - m_{s-2}(\phi_1 \dots \phi_1) \cdot \phi_1^2 \\
+ \dots + (-1)^{s-1} m_2(\phi_1 \phi_1) \cdot \phi_1^{s-2} + (-1)^s m_1 \phi_1 \cdot \phi_1^{s-1} + (-1)^{s+1} \cdot \phi_1^s]
\end{aligned}$$

and

$$m_s(\phi_1 \dots \phi_i \dots \phi_j \dots \phi_s) = m_s(\phi_1 \dots \phi_j \dots \phi_i \dots \phi_s) \qquad i, j = 1 \dots s$$

We arrive at the formulæ directly by differentiating

$$s!\, m_s(\phi_1, \phi_1 \cdots \phi_1) = \begin{vmatrix}
m\phi_1 & 1 & 0 & 0 & 0 & \dots & 0 \\
m\phi_1^2 & m\phi_1 & 2 & 0 & 0 & \dots & 0 \\
m\phi_1^3 & m\phi_1^2 & m\phi_1 & 3 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \\
m\phi_1^{s-1} & m\phi_1^{s-2} & m\phi_1^{s-3} & \dots & \dots & \dots & s-1 \\
m\phi_1^s & m\phi_1^{s-1} & m\phi_1^{s-2} & \dots & \dots & \dots & m\phi_1
\end{vmatrix}$$

103. Theorem. A study of the structure of all algebras of the Scheffers type gives us the structure of all algebras of the Cartan type, as we may produce any Cartan algebra by substituting for each partial modulus of the Scheffers type a quadrate, and then substitute for each unit of the algebra a sub-algebra consisting of the product of this unit by the two quadrates which correspond to its characters.[1]

104. Theorem. Each Scheffers algebra may be deduced from a Peirce algebra by breaking the modulus up into partial moduli, accompanied by corresponding separations of the units. For, if all partial moduli of a Scheffers algebra are deleted from the algebra, leaving only the modulus, and a set of nilpotent units, we have a Peirce algebra. Any Peirce algebra may be considered to have been produced in this manner, so that to any Scheffers algebra corresponds a Peirce algebra, and to any Peirce algebra correspond a number of Scheffers algebras.

105. Theorem. If the characteristic function of an algebra be

$$\Delta_1^{\mu_1} \dots \Delta_p^{\mu_p} = 0$$

wherein Δ_i is a determinant in which ζ, the general number of the algebra, occurs only on the diagonal, and the other constituents are linear homogeneous functions of the coordinates of ζ, and if we substitute for ζ where it occurs ψ,

[1] CARTAN 2. Cf. MOLIEN 1; SHAW 4.

any arbitrary number of the algebra, then the resulting expression may be written $C(\psi) \equiv \Delta_1^{\mu_1}(\psi)\,\Delta_2^{\mu_2}(\psi)\,\ldots\,\Delta_p^{\mu_p}(\psi)$. This expression will vanish only for

$$\psi = \zeta_1 \quad K^1\zeta_1 \ldots K^{\mu_1-1}\zeta_1 \ldots \zeta_i \quad K^1\zeta_i \ldots K^{\mu_i-1}\zeta_i \ldots$$

wherein $K^j\zeta_i$ has the meaning given in part II, chapter XIX, art. 3.

Thus the algebra whose characteristic equation is

$$\begin{vmatrix} x'_{00}e_0 - \zeta & x'_{01} \\ x'_{10} & x'_{11}e_0 - \zeta \end{vmatrix} (x''_0 e_0 - \zeta) = 0$$

gives the expression

$$\begin{vmatrix} x'_{00} - \psi & x'_{01} \\ x'_{10} & x'_{11} - \psi \end{vmatrix} (x''_0 - \psi) \equiv C(\psi)$$

This expression vanishes when and only when $\psi = q_1$, Kq_1, or q_2; wherein

$$\begin{aligned} q_1 &= x'_{00}\lambda_{11} + x'_{11}\lambda_{22} + x'_{01}\lambda_{12} + x'_{10}\lambda_{21} \\ Kq_1 &= x'_{11}\lambda_{22} + x'_{00}\lambda_{11} + x'_{01}\lambda_{21} + x'_{10}\lambda_{12} \\ q_2 &= x_0\lambda_{33} \end{aligned}$$

That is, the expression is factorable into $(\psi - q_1)(\psi - Kq_1)(\psi - q_2)$.

As a corollary, the expression

$$\begin{vmatrix} a_{00} - \theta & a_{01} & \ldots a_{0\,n-1} \\ a_{10} & a_{11} - \theta & a_{1\,n-1} \\ \cdot & \cdot & \cdot \\ a_{n-1\,0} & a_{n-1\,1} & a_{n-1\,n-1} - \theta \end{vmatrix}$$

is factorable in the matric range of q_1, $Kq_1 \ldots K^{n-1}q_1$.

IV. ASSOCIATIVE UNITS.

106. Definition. The multiplication formula in § 100 may be used to introduce certain useful new conceptions. It reads

$$\zeta \cdot \xi_{jk} = \Sigma a_{j'j\,k'-k}\,\xi_{j'k'}$$

$$k' - k \geqq 0 \qquad \mu_{j'} - k > k' - k \geqq \mu_{j'} - \mu_j \qquad j' \leqq j \text{ if } k' = k$$

Let us consider an algebra made up of units which will be called *associative units*, represented by λ_{rst}, such that

$$\lambda_{ijk} \cdot \xi_{j'k'} = c\vartheta_{jj'} \cdot \xi_{i\,k'+k}$$

where

$$\begin{aligned} &c = 1 \text{ if } \mu_i > k \geqq \mu_i - \mu_{j'} \qquad k \geqq 0 \qquad i > j \text{ if } k = 0 \\ &c = 0 \text{ if } \mu_i \leqq k < \mu_i - \mu_{j'} \end{aligned}$$

Since there is a modulus e_0, and since $\xi_{j'k'} = \xi_{j'k'}e_0$, every unit $\xi_{j'k'}$ is expressible as a sum of these units λ_{ijk} multiplied by proper coefficients, and every number ζ is expressible as a sum of the units with proper coefficients. Hence, we may express ζ in the form

$$\zeta = \Sigma \,.\, a_{ijk}\,\lambda_{ijk}$$

$$\mu_i > k \geqq \mu_i - \mu_j \qquad k \geqq 0 \qquad i > j \text{ when } k = 0$$

The Peirce algebra is expressible therefore as a sub-algebra of the algebra of the associative units whose laws of multiplication[1] are

$$\lambda_{ijk}\,\lambda_{i'j'k'} = \vartheta_{ji'}\,c\lambda_{ij'\,k+k'}$$

where

$$\mu_i > k \geqq \mu_i - \mu_j \qquad k \geqq 0 \qquad i > j \text{ when } k = 0$$
$$\mu_i > k' \geqq \mu_i - \mu_{j'} \qquad k' \geqq 0 \qquad i > j' \text{ when } k' = 0$$

$$c = 1 \text{ if } \mu_i > k + k' \geqq \mu_i - \mu_{j'} \qquad k + k' \geqq 0 \qquad i > j' \text{ when } k + k' = 0$$
$$c = 0 \text{ if } \mu_i \leqq k + k' < \mu_i - \mu_{j'}$$

107. Definition. An expression of an algebra in terms of associative units will be called a *canonical* expression. In many cases the associative units are the units of the algebra, in part at least, but the units of the algebra will frequently occur as irreducible sums of these units with certain parametric coefficients. This theorem extends C. S. Peirce's theorem that every linear associative algebra is a sub-algebra of a quadrate[2] of order r^2.

108. Theorem. The Scheffers algebras derived from this Peirce algebra have partial moduli of the form

$$e_i = \Sigma\lambda_{j_i j_i 0} \qquad\qquad j_i = 1 \ldots\ldots g_i$$

When each partial modulus e_i is of the form λ_{ii0}, the Scheffers algebra coincides with the algebra of which the Peirce algebra is a sub-algebra. Such Scheffers algebras will be called *primary algebras.* The units in any Scheffers algebra are separable into classes according to their characters, those of character j having in their expression units λ of the type

$$\lambda_{sj_ik} \text{ or } \lambda_{j_isk} \qquad\qquad j_i = 1 \ldots\ldots g_i$$

109. Definition. The units of a Scheffers algebra are separable into those of characters, $(\alpha\alpha)$, and those of characters $(\alpha\beta)$, $\alpha \neq \beta$. Those of characters $(\alpha\alpha)$ constitute the *direct* units. Those of characters $(\alpha\beta)$ are the *skew* units.[3]

110. Theorem. The pre-latent (post-latent) equation must contain the factor $(a_{ii0} - \zeta)$ to that power which is the sum of the multiplicities belonging to i:

$$\mu_1^{(i)} + \mu_2^{(i)} + \ldots\ldots \mu_{c_i}^{(i)}$$

The characteristic equation will contain $(a_{ii0} - \zeta)$ to that power which equals the maximum multiplicity[4] $\mu_1^{(i)}$.

111. Theorem. A Cartan algebra will have for a canonical expression

$$\zeta = \Sigma\, a_{ijk}^{(\alpha\beta)}\, \lambda_{\alpha\beta 0}\, \lambda'_{ijk}$$

where the units λ and λ' are independent of each other.

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[1] Shaw 4, 5. [2] C. S. Peirce 1, 4. [3] Scheffers 3. [4] Scheffers 3; Shaw 4.

112. **Theorem.** We may obviously combine these forms into still more compound expressions as

$$\zeta = \Sigma\, a_{(i_1 j_1 k_1)(i_2 j_2 k_2)} \ldots \lambda'_{i_1 j_1 k_1} \lambda''_{i_2 j_2 k_2} \ldots$$

Such numbers are evidently associative, and could be considered to be the symbolic product of algebras with only one λ.

113. **Theorem.** Returning to the equations of the algebra §108, we see they evidently depend on those associative units which are of weight zero. The equations are

$$\textit{characteristic:} \quad 0 = \Delta \,.\, \zeta = \Delta_1^{\mu_{11}} \Delta_2^{\mu_{21}} \ldots \Delta_p^{\mu_{p1}}$$

$$\textit{pre-latent:} \quad 0 = \Delta' . \zeta = \prod_{i=1}^{p} . \Delta_i^{\sum_{j=1}^{p} n_{ij} w_j}$$

$$\textit{post-latent:}^{1} \quad 0 = \Delta'' . \zeta = \prod_{i=1}^{p} . \Delta_i^{\sum_{j=1}^{p} n_{ji} w_j}$$

114. **Theorem.** The number $\Delta_1(\zeta)$ can not contain any associative unit of the form $\lambda_{j_1 j_1 0}$, where the constituents of Δ_1 are of the form[2] $a_{j_1 j_1 0}$, $j_1 = 1 \ldots g_1$. The factor $\Delta_i(\zeta)$ is the i-th *shear* factor of ζ.

115. **Theorem.** The product $\Delta_1 \zeta \,.\, \Delta_2 \zeta$ can not contain any associative unit of the form $\lambda_{j_1 j_1 0}$, or $\lambda_{j_2 j_2 0}$. The theorem may be extended to the product of any number of shear factors.[2]

116. **Theorem.** The product $(\Delta_1 \zeta)^m$ can not contain any associative unit of the forms

$$\lambda_{j_1 j_1 0},\ \lambda_{j_1 j_1 1} \ldots \lambda_{j_1 j_1\, m-1}$$

117. **Theorem.** The third subscript in λ_{ijk}, k, is called the *weight* of λ. Every number ζ may be written in the form

$$\zeta = \zeta^{(a)} + \zeta^{(b)} + \ldots + \zeta^{(m)} \qquad a \geqq 0,\ b > a$$

The weight of ζ is the weight a of its lowest term. The weight of the product of two numbers is the sum of their weights.

118. **Theorem.** The terms $\zeta^{(0)}$ constitute an algebra. This may be called a *companion* algebra, and may or may not be a sub-algebra of the given algebra.[3] The quadrate units of an algebra evidently belong also to the companion algebra.

119. **Theorem.** To every transformation of the units of a companion algebra corresponds a transformation of the units of the given algebra. Hence

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[1] Cartan 2. [2] Shaw 4.

[3] Cf. Molien 1. "Begleitende" systems include these companion algebras, and may or may not be sub-algebras of the given algebra.

the $\zeta^{(0)}$ terms may always be taken according to the simplest form for the companion algebra.[1]

120. Theorem. If the general equation of an algebra is

$$\zeta^r - m_1 \zeta^{r-1} + m_2 \zeta^{r-2} \ldots . = 0$$

and if when $\zeta = \sum_{i=1}^{r} x_i e_i$ we put $\nabla = \sum_{i=1}^{r} . e_i \frac{\partial}{\partial x_i}$, then $\nabla . m_2 = 0$ gives r equations, not necessarily independent, from which the r coordinates may be expressed linearly in terms of r_1 arbitrary numbers. These determine the nilpotent system; or from the $r - r_1$ coordinates which vanish, the DEDEKIND sub-algebra.[2]

121. Theorem. Since $\nabla = \zeta \, I\zeta \, \nabla$, and $I\zeta \nabla \cdot \rho = \zeta$, therefore

$$\nabla \cdot m_1(\rho) = \nabla \cdot I\zeta_1(\rho\zeta_1) = \zeta_2 . I\zeta_2 \nabla \cdot I\zeta_1(\rho\zeta_1) = \zeta_2 I\zeta_1(\zeta_2\zeta_1)$$

But $I . \zeta_1(\zeta_2\zeta_1) = m_1(\zeta_2)$, therefore we have

$$\nabla m_1(\rho) = \zeta_2 m_1(\zeta_2)$$

This can vanish only if

$$m_1(e_i) = 0 \qquad i = 1 \ldots . r$$

Again,

$$m_2(\rho) = I\zeta_1 A\zeta_2 A(\rho\zeta_1)(\rho\zeta_2)$$

hence

$$\begin{aligned}
\nabla m_2(\rho) &= \zeta_3 I\zeta_1 A\zeta_2 A[(\zeta_3\zeta_1)(\rho\zeta_2) - (\zeta_3\zeta_2)(\rho\zeta_1)] \\
&= 2\zeta_3 \begin{vmatrix} I\zeta_1(\zeta_3\zeta_1) & I\zeta_1(\rho\zeta_2) \\ I\zeta_2(\zeta_3\zeta_1) & I\zeta_2(\rho\zeta_2) \end{vmatrix} \\
&= 2 \sum_{i=1}^{r} e_i \begin{vmatrix} I\zeta_1(e_i\zeta_1) & I\zeta_1(\rho\zeta_2) \\ I\zeta_2(e_i\zeta_1) & I\zeta_2(\rho\zeta_2) \end{vmatrix} \\
&= 2 \sum_{i=1}^{r} e_i [I . \zeta_1(e_i\zeta_1) I\zeta_2(\rho\zeta_2) - I\zeta_2(e_i\rho\zeta_2)] \\
&= 2 \sum_{i=1}^{r} e_i [m_1(e_i) . m_1(\rho) - m_1 . (e_i \rho)]
\end{aligned}$$

This vanishes if, and only if,

$$I\zeta_1(e_i\zeta_1) I\zeta_2(\rho\zeta_2) - I\zeta_2(e_i\rho\zeta_2) = 0 \qquad i = 1 \ldots . r$$

or

$$\sum_{j=1}^{r} x_j \{m_1(e_i) m_1(e_j) - m_1(e_i e_j)\} = 0 \qquad i = 1 \ldots . r$$

These are the equations referred to in §120. The method used here has an obvious extension.

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[1] Cf. SHAW 5. [2] CARTAN 2.

V. SUB-ALGEBRAS. REDUCIBILITY. DELETION.

122. **Definition.** A sub-algebra consists of the totality of numbers ζ such that

$$\zeta = \Sigma x_i e_i \qquad i = 1 \ldots . r', r' < r$$

for which[1]

$$\zeta_1 \zeta_2 = \Sigma . x_i' x_j'' \gamma_{ijk} e_k \qquad i, j, k = 1 \ldots . r'$$

123. **Theorem.** In a Scheffers algebra all units with like pre- and post-character $(\alpha\alpha)$ define a Peirce sub-algebra.[2]

124. **Theorem.** The Peirce sub-algebras formed according to § 123 define together the *direct* sub-algebra. The characteristic equation of this sub-algebra does not differ from the equation of the algebra.[3]

125. **Theorem.** The quadrates form a sub-algebra, the semi-simple system of CARTAN,[4] called a DEDEKIND algebra.[5]

126. **Theorem.** All units in a Cartan algebra with characters chosen from a single quadrate form a sub-algebra, the product of the quadrate by a Peirce algebra. Its equation has but one shear factor.

127. **Theorem.** All sub-algebras of § 126 determined by the different quadrates form the *direct* quadrate sub-algebra. Its equation does not differ from that of the algebra.

128. **Theorem.** All numbers which do not contain quadrate units form a sub-algebra called the nil-algebra (Cartan's pseudo-nul invariant system).[4] The units of this system are determinable to a certain extent (viz. those which also belong to the direct sub-algebra of § 127) from the equation of the algebra. The other units are not determinable from the characteristic equation of the algebra.[3]

129. **Definitions.** All numbers ζ, which are expressible in the form

$$\zeta = \sum_{i=1}^{r'} x_i e_i \qquad r' \leqq r$$

form a *complex*. The entire complex may be denoted by E_1, E_2, etc., $E \equiv E_0$ denoting the original algebra.[6]

The product of two complexes consists of the complex defined by the products of all the units defining E_1 into the units defining E_2, indicated[7] by

$$E_1 . E_2$$

An algebra E is *reducible* when its numbers may all be written in the

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[1] On the general subject see STUDY 1, 2, 3; SCHEFFERS 1, 2, 3, 4, 7; B. PEIRCE 1, 3; HAWKES 1. 2.
[2] SCHEFFERS 3. CARTAN 2. [3] SHAW 4. [4] CARTAN 2. [5] FROBENIUS 14.
[6] EPSTEEN and WEDDERBURN 2. [7] EPSTEEN and WEDDERBURN 2; FROBENIUS 11.

form $\zeta = \zeta_1 + \zeta_2$ where ζ_1 belongs to a complex E_1, ζ_2 to a complex E_2, such that,[1]

$$E_1 \cdot E_1 = E_1 \qquad E_1 \cdot E_2 = 0 \qquad E_2 \cdot E_1 = 0 \qquad E_2 \cdot E_2 = E_2$$

An algebra is *irreducible* when it can not be broken up in this way. When reducible into a complexes we may write

$$E = E_1 + E_2 + \ldots\ldots + E_a$$

130. Theorem. An algebra is reducible into irreducible sub-algebras in only one way.[2]

131. Theorem. The necessary and sufficient condition of reducibility is the presence of h numbers $e_1 \ldots\ldots e_h$, such that if ζ is any number,[2]

$$\zeta e_\alpha = e_\alpha \zeta \qquad e_\alpha^2 = e_\alpha \qquad e_\alpha e_\beta = e_\beta e_\alpha = 0 \qquad \alpha = 1 \ldots\ldots h,\ \alpha \neq \beta$$

132. Theorem. The characteristic function of a reducible algebra is the product of the characteristic functions of its irreducible sub-algebras.[2] The order is the sum of the orders of the sub-algebras, and the degree is the sum of the degrees of the sub-algebras.

133. Definitions. The region common to two regions, or the complex common to two complexes E_1, E_2, is designated by E_{12}. If the complex E_1 is included in the complex E_2 this will be indicated by[3] $E_1 \leqq E_2$.

The reducibility used by B. Peirce is defined thus, E is reducible[4], if

$$E = E_1 + E_2 \qquad E_1^2 \leqq E_1 \qquad E_2^2 \leqq E_2 \qquad E_1 E_2 \leqq E_{12} \qquad E_2 E_1 \leqq E_{12}$$

An algebra is *deleted* by a complex E_2 if the units in E_2 are erased from all expressions of the algebra, including products. The result is a *delete* algebra, if it is *associative*. It may not contain a modulus however.[5]

134. Theorem. Let the product of $\zeta\sigma$ be given by the equation

$$\zeta\sigma = \sum_{i,j,k}^{1\ldots\ldots r} x_i y_j \gamma_{ijk} e_k = \sum_{k=1}^{r} x_k' e_k$$

If the units may be so transformed that the product may be expressed by means of the equations

$$x_i' = \sum_{j,k}^{1\ldots\ldots r'} x_j y_k \gamma_{ijk} \qquad i = 1 \ldots\ldots r' \qquad r' < r$$

$$x_{i'}' = \sum_{j,k}^{1\ldots\ldots r} x_j y_k \gamma_{i'jk} \qquad i' = r' + 1 \ldots\ldots r$$

then the units $e_1 \ldots\ldots e_{r'}$, define a *delete* algebra,[6] called hereafter a MOLIEN algebra. If an algebra has no MOLIEN algebra, it is *quadrate.*

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[1] See references § 122. [2] SCHEFFERS 3, 4. [3] EPSTEEN and WEDDERBURN 2.

[4] EPSTEEN and WEDDERBURN 2. On the definitions of reducibility see EPSTEEN and LEONARD 3; LEONARD 2.

[5] SCHEFFERS 3, 4; HAWKES 1, 3. Cf. MOLIEN 1; SHAW 5.

[6] MOLIEN 1. This is Molien's "begleitendes" system.

135. Theorem. A MOLIEN algebra of a MOLIEN algebra is a MOLIEN algebra of the original algebra. Two MOLIEN algebras which are such that the coordinates of the numbers of the two algebras have q linear relations, *i.e.*, whose numbers are subject to q linear relations, possess a common MOLIEN algebra of order q, and conversely. If the MOLIEN algebras of an algebra have no common MOLIEN algebras, then the numbers in the different MOLIEN algebras are linearly independent.[1]

136. Theorem. If the complex of the linearly independent numbers of the form $\zeta\sigma - \sigma\zeta$ be deleted from an algebra, the remaining numbers form a commutative algebra.[1]

137. Theorem. If the commutative algebra of § 136 contains but one unit the original algebra is a quadrate.[1]

138. Theorem. If the delete algebra in § 136 contain more than one unit it may be further deleted until the delete contains but one unit. This unit will belong to a quadrate algebra which is a delete of the original algebra.[1]

139. Theorem. The scalar of any number contains only coordinates which belong to the units in the commutative delete algebra.[1]

140. Theorem. The pre- and post-latent functions of a delete algebra are factors of the corresponding equations of the original. The characteristic equation of the delete is a factor of the characteristic equation of the original.[1]

141. Theorem. The two equations of a quadrate delete algebra are powers of the same irreducible expression.[1]

142. Theorem. An algebra is a quadrate if its characteristic equation is irreducible and if the scalar of any number contains only coordinates belonging to the units of the quadrate (which may be a delete algebra).[1]

143. Theorem. The irreducible factors of the characteristic equation of an algebra are the characteristic functions of its delete quadrate algebras.[1]

144. Theorem. The number of units of a delete quadrate is the square of the order m, of its characteristic equation. If they are e_{ij}, then

$$e_{ij}\,e_{kl} = \vartheta_{jk}\,e_{il} \qquad i, j, k, l = 1 \;\ldots\; m$$

The delete quadrate is also a sub-algebra of the original.[2]

145. Theorem. If, in a Scheffers algebra, the product of ζ into and by the units $e_r, e_{r-1} \ldots e_{r-r_1}$, vanishes, provided ζ is not a modulus or a partial modulus, then the algebra may be deleted by the complex of $e_r \ldots e_{r-r_1}$. The

[1] MOLIEN 1.

[2] MOLIEN 1. Molien points out that the units may be classified according to their quadrate character, thus approaching Cartan's theorem, § 99.

delete algebra will have an equation with all the factors of the original algebra, but each appearing with an exponent less by unity for each deleted direct unit belonging to the factor.[1]

146. Definition. The *deficiency* of a Peirce algebra is the difference between its order and its degree.[2]

147. Theorem. The units of a Peirce algebra may be so chosen that, if it is of deficiency δ, one unit may be deleted, giving a delete algebra of deficiency $\delta - 1$, which is a sub-algebra of the original.[2]

148. Definitions. An algebra E is *semi-reducible of the first kind* when it consists of two complexes, E_1, E_2 such that,[3]

$$E_1 E_1 \leqq E_1 \qquad E_1 E_2 \leqq E_2 \qquad E_2 E_1 \leqq E_2 \qquad E_2 E_2 \leqq E_2$$

An algebra is *semireducible of the second kind* when it satisfies the equations[4]

$$E_1 E_1 \leqq E_1 \qquad E_1 E_2 \leqq E_2 \qquad E_2 E_1 = 0 \qquad E_2 E_2 \leqq E_2$$

If in any algebra

$$E_1 E_1 \leqq E \qquad E_1 E_2 \leqq E_2 \qquad E_2 E_1 \leqq E_2 \qquad E_2 E_2 \leqq E_2$$

then E_2 is called an *invariant sub-algebra.*[5]

149. Theorem. If E has an invariant sub-algebra E_2, the algebra K produced by deleting E_2 is a delete of E, called *complementary* to[5] E_2.

150. Theorem. If E_1 is a maximal invariant sub-algebra of E, and if there exists a second invariant sub-algebra E_2 of E, then either E is reducible or E_2 is a sub-algebra[5] of E_1.

151. Theorem. If E_1 and E_2 are maximal invariant sub-algebras of E, and if $E_{12} \neq 0$, then E_{12} is a maximal invariant sub-algebra of both E_1 and E_2.[5]

152. Theorem. A *normal* series of sub-algebras of E, is a series $E_1, E_2, \ldots.$ such that E_s is a maximal invariant sub-algebra of E_{s-1} $(E_0 \equiv E)$. If $K_1, K_2, \ldots.$ are the corresponding complementary deletes, then apart from the order the series $K_1, K_2, \ldots.$ is independent of the choice of $E_1, E_2, \ldots.$[5]

153. Theorem. Let a_s be the order of E_s; l_s, the difference between a_{s-1} and the maximal order of a sub-algebra of E_{s-1} which contains E_s; $k_s = a_{s-1} - a_s$. Then the numbers $l_1, l_2, \ldots.$ are independent of the choice of the normal series apart from their order. A like theorem holds for $k_1, k_2, \ldots.$[5]

154. Definition. An algebra which has no invariant sub-algebra is *simple.*[5]

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[1] Scheffers 3. [2] Starkweather 1. [3] Epsteen 1, 2.
[4] Epsteen 1. [5] Epsteen and Wedderburn 2.

155. **Theorem.** The complementary deletes $K_1, K_2, \ldots$ are all simple.[1]

156. **Definition.** The series $E, P_1, P_2, \ldots$ is a *chief or principal series* when P_s is a maximal sub-algebra of P_{s-1} which is invariant[1] in E.

157. **Theorem.** The system of indices of composition is independent of the choice of the chief series, apart from the sequence.[1]

158. **Theorem.** An algebra is irreducible if its quadrates may be so arranged $Q_1, Q_2 \ldots Q_p$ that there are skew units of characters $(21), (32) \ldots (p\,p-1)$.[2]

VI. DEDEKIND AND FROBENIUS ALGEBRAS.

159. **Definition.** A DEDEKIND algebra is one which is the sum of quadrates $Q_1, Q_2 \ldots Q_h$. Its order[3] is $r = w_1^2 + \ldots + w_h^2$.

160. **Theorem.** A DEDEKIND algebra has a sub-algebra of order h, whose numbers are commutative with all numbers of the DEDEKIND algebra. No other numbers than those of this sub-algebra are so commutative.[4]

161. **Theorem.** A DEDEKIND algebra is reducible and the sub-algebras are found by multiplying by the numbers e_α, $\alpha = 1 \ldots h$, in terms of which the commutative sub-algebra may be defined. $[e_\alpha e_\beta = \vartheta_{\alpha\beta} e_\alpha]$.[5]

162. **Theorem.** The characteristic equation of a DEDEKIND algebra is $\Delta_1 \Delta_2 \ldots \Delta_h = 0$. The pre- and post-equations[5] are $\Delta_1^{w_1} \Delta_2^{w_2} \ldots \Delta_h^{w_h} = 0$.

163. **Theorem.** If a DEDEKIND algebra has only linear factors in its equation it is a commutative algebra.[5]

164. **Theorem.** The scalar of e_α is given by the equation

$$S \,.\, e_\alpha = \frac{w_\alpha^2}{r}$$

The scalar within a single quadrate, Q_i, may be indicated by S_i. For any number we have[5]

$$S \,.\, \zeta = \sum_i S_i \,.\, \zeta \qquad i = 1 \ldots h$$

165. **Theorem.** An algebra is a DEDEKIND algebra when in the general equation, m_2, the coefficient of ζ^{r-2}, contains each coordinate in such a way that the equations

$$\frac{\partial m_2}{\partial x_i} = 0 \qquad i = 1 \ldots r$$

give[6]

$$x_1 = \ldots = x_r = 0$$

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[1] EPSTEEN and WEDDERBURN 2.
[2] SCHEFFERS 3, 4.
[3] Cf. FROBENIUS 14. CARTAN 2. This is Cartan's semi-simple algebra.
[4] FROBENIUS 14. He calls these *invariant* numbers.
[5] FROBENIUS 14.
[6] CARTAN 2, see § 121. Evidently $| m_2(e_i, e_j) | \neq 0$.

166. Theorem. If Δ_i is the determinant shear factor corresponding to the quadrate Q_i, then $S_i \,.\, \Delta_i = 0$ for all numbers of the algebra, and if e_i is the partial modulus of this quadrate,[1]

$$e_i \Delta_i = \Delta_i e_i = 0$$

The $i + 1$ scalar coefficient of any numbers vanishes; i. e.

$$m_{i+1}^{(i)}(\xi_1, \ldots \xi_i) = 0$$

167. Theorem. If $\Delta_i(a) = \Delta_i(b)$ then for a determinate number[2] c

$$c^{-1}ac = b$$

168. Definition. A Frobenius algebra is one which can be defined by r numbers $o_1 \ldots o_r$ which satisfy the equations

$$o_i^{m_i} = e_0 = o_1 \qquad i = 1 \ldots r$$

$$o_i o_j = o_k \qquad o_i^{-1} o_k = o_j \qquad o_i = o_k o_j^{-1} \qquad i, j = 1 \ldots r$$

$$o_i o_j \,.\, o_k = o_i \,.\, o_j o_k \qquad i, j, k = 1 \ldots r$$

The multiplication table of these units defines a *group*, and any group of finite order or infinite order may be made isomorphic to a Frobenius algebra.[3]

169. Definition. Two units o_i, o_j are *conjugate* if for some determinable unit o_k,

$$o_i = o_k o_j o_k^{-1}$$

If we operate on o_j by all units of the algebra, $o_1 \ldots o_r$, we arrive at r_t different units as results. These are said to constitute the t-th *conjugate class*. There will be k of these classes. Also r_t is a divisor of r.

170. Theorem. For each unit in a conjugate class we have (as o_j is the modulus or not):

$$S \,.\, o_k o_j o_k^{-1} = S \,.\, o_j = 1 \text{ or } 0$$

171. Theorem. If the sum of all the units in the t-th conjugate class be K_t, then for any unit

$$K_t o_i = o_i K_t \qquad i = 1 \ldots r$$

There are k different numbers K_t, $K_1 \ldots K_k$.

172. Theorem. The k numbers K_t, $t = 1 \ldots k$ constitute a commutative algebra of k dimensions, that is

$$K_t K_u = K_u K_t = \sum_{v=1}^{k} C_{tuv} K_v \qquad t, u = 1 \ldots k$$

173. Theorem. We have (according as o_t is not or is the modulus):

$$S \,.\, K_t = \frac{r_t}{r} S \,.\, \Sigma o_i o_t o_i^{-1} = r_t S \,.\, o_t = 0 \text{ or } r_t$$

174. Theorem. A Frobenius algebra is a Dedekind algebra of k quadrates. The k numbers K_t determine the k partial moduli, one for each quadrate.

[1] Frobenius 14. Shaw 4. [2] Frobenius 14. Other theorems appear in Chapter XIX, Part II.
[3] Frobenius 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13; Dickson 1, 2, 3, 4; Burnside 1, 5; Poincaré 4; Shaw 6.

The widths of the quadrates being represented by w_i, $i = 1 \ldots . k$, we have

$$K_t = \Sigma\, g_{ti}\, e_i \qquad i, t = 1 \ldots . k$$

175. Theorem. It follows that if we take scalars

$$r \,.\, SK_1 = r = w_1^2 + w_2^2 \ldots . + w_k^2$$
$$r \,.\, SK_j = 0 = w_1^2 g_{j1} + w_2^2 g_{j2} \ldots . + w_k^2 g_{jk} \qquad j = 2 \ldots . k$$

176. Theorem. Let the scalar of o_t in the quadrate i be represented by $S^{(i)}_{.} o_t$, or $s_t^{(i)}$, then

$$S \,.\, o_t = \sum_{i=1}^{k} w_i^2 \,.\, S^{(i)}_{.}\, o_t = \sum_{i=1}^{k} w_i^2 s_t^{(i)}$$

177. Theorem. We have

$$S^{(i)}_{.}\, K_t = g_{ti}\, S^{(i)}_{.}\, e_i = g_{ti} = r_t \,.\, S^{(i)}_{.}\, o_t = r_t\, s_t^{(i)}$$

Hence

$$r = w_1^2 s_1^{(1)} + w_2^2 s_1^{(2)} + \ldots . + w_k^2 s_1^{(k)}$$
$$0 = w_1^2 s_j^{(1)} + w_2^2 s_j^{(2)} + \ldots . + w_k^2 s_j^{(k)} \qquad j = 2 \ldots . k$$

If we write for $w_i\, s_j^{(i)}$ the symbol $\chi_j^{(i)}$, (called by Frobenius the i-th *characteristic* of o_j) we have

$$w_j = r\,\frac{\Delta_j}{\Delta} = \chi_1^{(i)}$$

where Δ is the determinant $|\chi_1^{(1)}, \chi_2^{(2)} \ldots . \chi_k^{(k)}|$ and Δ_j is the minor (including sign) of w_j. This determinant Δ evidently cannot vanish.

Also

$$K_t = r_t \sum_{i=1}^{k} e_i\, \frac{\chi_t^{(i)}}{w_i} = r_t\, \Sigma\, e_i\, s_t^{(i)}$$

and

$$(\Sigma K_t)^2 = r \Sigma K_t$$

178. Theorem. Hence

$$e_i = \begin{vmatrix} s_1^{(1)} & K_1/r_1 \ldots . & s_1^{(k)} \\ s_2^{(1)} & K_2/r_2 \ldots . & s_2^{(k)} \\ \ldots . & \ldots . & \\ s_k^{(1)} & K_k/r_k \ldots . & s_k^{(k)} \end{vmatrix} \div \begin{vmatrix} s_1^{(1)} & s_1^{(2)} & \ldots . & s_1^{(k)} \\ s_2^{(1)} & s_2^{(2)} & \ldots . & s_2^{(k)} \\ \ldots . & & \ldots . & \\ s_k^{(1)} & s_k^{(2)} & \ldots . & s_k^{(k)} \end{vmatrix}$$

$$e_i = \frac{w_i \left| \chi_1^{(1)}\, \chi_2^{(2)} \ldots . \dfrac{K_i}{r_i} \ldots . \chi_k^{(k)} \right|}{\Delta}$$

179. Theorem. For all values of a, b

$$\sum_{d=1}^{r} S^{(i)} \,.\, o_a\, o_d\, o_b\, o_d^{-1} = \frac{r}{r_b} S^{(i)} \,.\, o_a\, K_b = r S^{(i)} \,.\, o_a\, S^i \,.\, o_b$$

or

$$r s_a^{(i)}\; s_b^{(i)} = \sum_{d=1}^{r} S^{(i)}\, o_a\, o_d\, o_b\, o_d^{-1}$$

or

$$\frac{r}{r_b} \chi_a^{(i)}\, \chi_b^{(i)} = w_i \sum_c{}' \chi_{ac}^{(i)}$$

where Σ'_c takes o_c over the r_b values in the conjugate class[1] of o_b.

Also[2] $w_i\, w_j\, s_a^{(i)}\, s_a^{(j)} = \sum_{u=1}^{k} d_{iju}\, s_a^{(u)}$ or $\chi_a^{(i)}\, \chi_a^{(j)} = \sum_{u=1}^{k} d'_{iju}\, \chi_a^{(u)}$

[1] See references to § 168. These apply to theorems following.

[2] Burnside 5.

180. Theorem. $\sum_{b=1}^{r} . S^{(i)} . o_a o_b^{-1} S^{(i)} o_b = \frac{r}{w_i^2} S^{(i)} o_a$

181. Theorem. $\sum_{b=1}^{r} . S^{(i)} . o_a o_b^{-1} S^{(j)} o_b = 0$ $\qquad i \ddagger j$

182. Theorem. $\sum_{b=1}^{r} S^{(i)} o_b . S^{(i)} o_b^{-1} = \frac{r}{w_i^2}$

183. Theorem. $\sum_{b=1}^{r} S^{(i)} o_a o_b^{-1} S^{(i)} o_c o_b = \frac{r}{w_i^2} S^{(i)} . o_a o_c$

184. Theorem. $\sum_{a,b=1}^{r} S^{(i)} . o_a^{-1} o_b^{-1} o_a o_b = \frac{r^2}{w_i^2}$

185. Theorem. If o_1 is an independent generator of the group of units, $o_0 \ldots . o_{r-1}$, and if we form the *t-th LaGrangian* of o_1, that is,

$$f_{1t} = \frac{1}{m_1} (1 + \omega^t o_1 + \omega^{2t} o_1^2 + \ldots . + \omega^{(m_1-1)t} o_1^{m_1-1})$$

where ω is a primitive m_1-th root of unity, and m_1 is the *order* of o_1 $(o_1^{m_1} = o_0)$ then for any number of the algebra, ζ, we have a product

$$\zeta f_{1t}$$

such that all numbers of the algebra are separable into m_1 mutually exclusive classes of the forms (where it is sufficient for ζ to be any one of the units o_j when the group is written in the form $o_j o_1^s$).

$$\zeta f_{1t} \qquad (t = 1 \ldots . m_1)$$

For ζf_{1t}, we have[1]

$$\zeta f_{1t} f_{1t} = \zeta f_{1t} \qquad \zeta f_{1t} . f_{1s} = 0 \qquad s \ddagger t$$

186. Theorem. If o_2 is a second independent generator, then we may determine the equations of $o_2 f_{1t}$ $(t = 1 \ldots . m_1)$. The latents Z_i, determined as in § 48, used as right multipliers, separate the numbers of the algebra into mutually exclusive classes, such that if these latents are f_{1tu}, then (if $u \ddagger u'$, $t \ddagger t'$)

$$\zeta f_{1tu} f_{1tu} = \zeta f_{1tu} \qquad \zeta f_{1tu} f_{1t'u'} = 0$$

This process of determining latents by the independent generators may be continued until they are in turn exhausted.

187. Theorem. The ultimate latents are scalar multiples of independent idempotents of the forms $\lambda_{ii0}^{(s)}$, where $i = 1 \ldots . w_i$; $s = 1 \ldots . k$. Multiplication right and left by these idempotents will determine every quadrate unit $\lambda_{ij0}^{(s)}$, $i, j = 1 \ldots . w_i$; $s = 1 \ldots . k$, in terms of the c generators $o_1 \ldots . o_c$.

188. These results may be extended easily to cases in which the coefficients of the units o_i are restricted to certain fields.

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[1] SHAW 6. This reference applies to §§ 186, 187.

VII. SCHEFFERS AND PEIRCE ALGEBRAS.

189. **Theorem.** Every Scheffers algebra with h partial moduli has h sub-algebras, each with like pre- and post- characters.

190. **Theorem.** The general equation of a Scheffers algebra of h partial moduli is of the form[1]

$$\prod_{i=1}^{h} \cdot (a_i - \zeta)^{n_{i1} + n_{i2} + \cdots + n_{ih}} = 0$$

191. **Theorem.** Every number of a Scheffers algebra satisfies the general equation of its direct sub-algebra, which is

$$\prod_{i=1}^{h} (a_i - \zeta)^{n_{ii}} = 0$$

This equation is the *intermediate* equation of the algebra.

192. **Theorem.** The characteristic equation of a Scheffers algebra is

$$\prod_{i=1}^{h} (a_i - \zeta)^{\mu_i} = 0$$

193. **Theorem.** A Peirce algebra may have its units taken in the form[2]

$$\xi_s \theta^t \qquad s = 1 \ldots p \quad t = 0 \ldots \mu_s - 1$$

194. **Theorem.** Units containing θ^t, $t > 0$, may be deleted, and the remaining numbers will then form a companion delete algebra, called the *base* of the Peirce algebra.[3]

195. **Theorem.** Any Peirce algebra may be made to serve as a base by expressing its units in terms of associative units of weight zero.[3]

196. **Theorem.** The product of two units follows the law[4]

$$\xi_{s'} \theta^{t'} \; \xi_{s''} \theta^{t''} = \Sigma b_{s' t' s'' t'' s''' t'''} \, \xi_{s'''} \theta^{t'''} \qquad t''' \geqq t' + t''$$

197. **Theorem.** A Peirce algebra of order r, degree r, is composed of the units[5]

$$e_1 = \lambda_{110} \qquad e_2 = \lambda_{111} \qquad e_3 = \lambda_{112} \ldots\ldots e_r = \lambda_{11\,r-1}$$

These have been called by Scheffers, Study algebras.

198. **Theorem.** A Peirce algebra of order r, degree $r - 1$, is composed of the units

$$e_1 = \lambda_{110} + \lambda_{220} \quad e_2 = \lambda_{210} + b\lambda_{12\,r-2} \quad e_3 = \lambda_{111} + a\lambda_{12\,r-2} \quad e_4 = \lambda_{112} \ldots e_r = \lambda_{11\,r-2}$$

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[1] Cartan 2. [2] Shaw 5. Cf. Strong 1. [3] Shaw 5. [4] Shaw 5. Cf. Scheffers 3; Cartan 2.
[5] B. Peirce 3; Scheffers 3; Hawkes 1; Shaw 5; Study 3.

This is reducible, if a and b do not vanish, to the case of $a = 1 = b$.

If $a = 0$, we may take $b = 1$ or 0. If $b = 0$, we may take $a = 1$ or 0.

When $r = 4$, either $a = 1, b$ has any value; or $a = 0$, $b = 1$; or $a = 0$, $b = 0$.

If $r = 3$, $a = 0$, $b = 0$.[1]

199. Theorem. A Peirce algebra of order r, degree $r - 2$, is of one of the following types.[2] Only the forms of e_0, e_1, e_2, e_3, e_4 are given since in every case

$$e_5 = \lambda_{113} \dots \, e_{r-2} = \lambda_{11\,r-4} \qquad e_{r-1} = \lambda_{11\,r-3}$$

The λ will be omitted in each case.

When $r > 6$.

(1). $e_0 = (110) + (220) + (330)$, type of algebra $(i, i^2, j, j^2 \dots j^{r-2})$

(11). $e_1 = (210) + (320) + (13\,r-2)$ $\quad e_2 = (310) + (12\,r-2)$ $\quad e_3 = (111)$ $\quad e_4 = (112)$

(12). $e_1 = (210) + (320)$ $\quad e_2 = (310)$ $\quad e_3 = (111)$ $\quad e_4 = (112)$

(13). $e_1 = (210) + (320) + (13\,r-2)$ $\quad e_2 = (310) + (12\,r-2)$ $\quad e_3 = (111) + 2(13\,r-2)$ $\quad e_4 = (112)$

(14). $e_1 = (210) + (320)$ $\quad e_2 = (310)$ $\quad e_3 = (111) + 2\,(13\,r-2)$ $\quad e_4 = (112)$

(2). $e_0 = (110) + (220)$, type of algebra $(i, j, ij, j^2 \dots j^{r-2})$

(21). $e_1 = (210) + (12\,r-3)$ $\quad e_2 = (211) + (12\,r-2)$ $\quad e_3 = (111) + (221)$ $\quad e_4 = (112)$

(22). $e_1 = (210) + (12\,r-3)$ $\quad e_2 = (211) + (12\,r-2)$ $\quad e_3 = (111) + (221) + 2\,(12\,r-2)$ $\quad e_4 = (112)$

(23). $e_1 = (210) + (12\,r-3)$ $\quad e_2 = (211) + (12\,r-2)$ $\quad e_3 = (111) + (221) + 2\,(12\,r-3) + 2c\,(12\,r-2)$ $\quad e_4 = (112) + 4\,(12\,r-2)$ $\quad c = 0$ if $r \ne 8$

(24). $e_1 = (210) + (12\,r-3) + (12\,r-2)$ $\quad e_2 = (211) - (12\,r-2)$ $\quad e_3 = (111) - (221) - 2\,(12\,r-3)$ $\quad e_4 = (112)$

(25). $e_1 = (210) + (12\,r-3)$ $\quad e_2 = (211) - (12\,r-2)$ $\quad e_3 = (111) - (221)$ $\quad e_4 = (112)$

(26). $e_1 = (210) + h\,(12\,r-2)$ $\quad e_2 = (211)$ $\quad h = 0$ or 1 if $r \ne 7$ $\quad e_3 = (111) + \frac{c}{2-c}(221) + 2\,(2-c)\,(12\,r-3)$ $\quad e_4 = (112) + 4\,(11\,r-2)$

(27). $e_1 = (210) + (12\,r-3)$ $\quad e_2 = (211) - (12\,r-3)$ $\quad e_3 = (111) - (221)$ $\quad e_4 = (112)$

(28). $e_1 = (210) + h\,(12\,r-2)$ $\quad e_2 = (211)$ $\quad e_3 = (111) + (221) + 2\,(12\,r-3)$ $\quad e_4 = (112)$ $\quad h = 0$ or 1 if $r \ne 7$

(29). $e_1 = (210) + (12\,r-2)$ $\quad e_2 = (211)$ $\quad e_3 = (111) + d\,(221)$ $\quad e_4 = (112)$

[1] B. Peirce 3; Scheffers 3; Shaw 5.

[2] Starkweather 1. Cf. Shaw 5.

(2α). $e_1 = (210)$ $e_2 = (211)$ $e_3 = (111) + d\,(221)$ $e_4 = (112)$
(2β). $e_1 = (210) + (221) + (12\,r-2)$ $e_2 = (211)$ $e_3 = (111)$ $e_4 = (112)$
(2γ). $e_1 = (210) + (221)$ $e_2 = (211)$ $e_3 = (111)$ $e_4 = (112)$
(2δ). $e_1 = (210)$ $e_2 = (211)$ $e_3 = (111) + 2(12\,r-2)$ $e_4 = (112)$
(2ε). $e_1 = (210) + (12\,r-2)$ $e_2 = (211)$ $e_3 = (111) + 2\,(12\,r-2)$ $e_4 = (112)$
(2ζ). $e_1 = (210) + (12\,r-2)$ $e_2 = (211)$ $e_3 = (111)$ $e_4 = (112)$
(2η). $e_1 = (210)$ $e_2 = (211)$ $e_3 = (111)$ $e_4 = (112)$

(3). $e_0 = (110) + (220) + (330)$, type of algebra, $(i, j, k, k^2 \ldots . k^{r-2})$
(31). $e_1 = (210) + (12\,r-2)$ $e_2 = (310)$ $e_3 = (111)$ $e_4 = (112)$
(32). $e_1 = (210)$ $e_2 = (310)$ $e_3 = (111)$ $e_4 = (112)$
(33). $e_1 = (210) + g\,(13\,r-2)$ $e_2 = (310) + (12\,r-2)$ $e_3 = (111)$ $e_4 = (112)$
(34). $e_1 = (210) + (13\,r-2)$ $e_2 = (310) + (12\,r-2)$
$e_3 = (111) + 2\,(12\,r-2) + 2\,(13\,r-2)$ $e_4 = (112)$
(35). $e_1 = (210) + (13\,r-2)$ $e_2 = (310) + (12\,r-2)$
$e_3 = (111) + 2\,(13\,r-2)$ $e_4 = (112)$
(36). $e_1 = (210) + (12\,r-2) - (13\,r-2)$ $e_2 = (310) + (12\,r-2)$ $e_3 = (111)$
$e_4 = (112)$
(37). $e_1 = (210) + (12\,r-2)$ $e_2 = (310)$ $e_3 = (111) + 2\,(13\,r-2)$ $e_4 = (112)$
(38). $e_1 = (210) + (12\,r-2)$ $e_2 = (310)$ $e_3 = (111) + 2\,(12\,r-2)$ $e_4 = (112)$
(39). $e_1 = (210)$ $e_2 = (310)$ $e_3 = (111) + 2\,(13\,r-2)$ $e_4 = (112)$

When $r = 4$, 5, *or* 6. These cases may be found in XX.

200. Theorem. A Scheffers algebra of degree $r-1$, which is not reducible, must consist of two Study algebras, with one skew unit connecting them.[1]

201. Theorem. A Scheffers algebra of degree $r-2$, which is not reducible, must consist of

(A) Three Study algebras, E_1, E_2, E_3, with skew units (12), (23);
(B) One Study algebra, and one algebra of deficiency unity, with one skew unit connecting them;
(C) Two Study algebras, joined (1) by two skew units (12) (12), or (2) joined by skew units (12), (21).

202. Theorem. A Peirce algebra whose degree is two, is determined as follows: for $m \leqq \frac{2\,r-1-\sqrt{8\,r-7}}{2}$ we may take

$$e_1, e_2 \ldots . e_m \equiv E_1, \text{ such that } EE_1 = E_1 E = 0$$

The remaining units are such that

$$e_{m+i}\, e_{m+j} = \Sigma \gamma_{m+i,\, m+j,\, k}\, e_k \qquad e_0 = \text{modulus},\ k = 1 \ldots . m$$

$$i \leqq r-m-1 \qquad j \leqq r-m-1 \qquad \gamma_{m+i,\, m+j,\, k} = -\gamma_{m+j,\, m+i,\, k}$$

or in brief[1]

$$E = E_1 + E_2,\ E_1^2 = 0,\ E_1 E_2 = E_2 E_1 = 0,\ E_2^2 \leqq E_1,\ \zeta_i \zeta_j = -\zeta_j \zeta_i$$

[1] Scheffers 3.

One class of Peirce algebras of degree two, and order r, may be constructed from the algebras of degree two and order less than r, by adjoining to the expressions for the algebra chosen for the base other terms as follows: let the units of the base be $e_i \ldots . e_j \ldots .$ written with weight zero, say e_{i0}, e_{j0}; then the adjoined unit (deleted unit) being $e_{r-1} = \lambda_{111}$, we have for new units

$$e_{i0}' = e_{i0} + a_{i2}\,\lambda_{121} + a_{i3}\,\lambda_{131} + \ldots .$$
$$e_{j0}' = e_{j0} + a_{j2}\,\lambda_{121} + a_{j3}\,\lambda_{131} + \ldots .$$

and $a_{ij} = - a_{ji}$ for all values of i, j.

The second and only other class involve units of forms $\lambda_{i11} + \ldots .$ and are given by

$$e_{i0}' = e_{i0} + a_{21}^{(i)}\,\lambda_{121} + \ldots . + a_{22}^{(i)}\,\lambda_{221} + \ldots .$$
$$e_{j0}' = e_{j0} + a_{21}^{(j)}\,\lambda_{121} + \quad \ldots \quad + a_{22}^{(j)}\,\lambda_{221} + \ldots .$$
$$e_s = \lambda_{111} - \delta_1\,\lambda_{221} - \delta_2\,\lambda_{331} \ldots . \qquad \delta_1, \delta_2 \ldots . = 0 \text{ or } 1$$

and $a_{jk}^{(i)} = - a_{ik}^{(j)}$ for all values[1] of i, j, k.

203. Theorem. A Scheffers algebra of order r, degree two, consists of two partial moduli $\lambda_{110} + \lambda_{220} + \ldots . + \lambda_{m_1 m_1 0}$ and $\lambda_{m_1+1, m_1+1, 0} + \ldots . + \lambda_{rr0}$, and $r - 2$ skew units as follows[2]

$$\lambda_{m_1+2\,10} \ldots . \lambda_{r10} \qquad \lambda_{2\,m_1+1\,0} \qquad \lambda_{3\,m_1+1\,0} \ldots . \lambda_{m_1\,m_1+1\,0}$$

204. The subject of the invariant equations of Peirce and Scheffers algebras is under consideration. Some particular cases are given later.

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[1] Shaw 5. [2] Scheffers 3.

VIII. KRONECKER AND WEIERSTRASS ALGEBRAS.

1. KRONECKER ALGEBRAS.

205. **Definition.** A commutative algebra is one such that every pair of numbers ζ_i, ζ_j in it, satisfy the equation :[1]

$$\zeta_i \zeta_j = \zeta_j \zeta_i$$

206. **Theorem.** An algebra is commutative when its units are commutative.

207. **Theorem.** The characteristic equation of a commutative algebra can contain only linear factors, if the coordinates belong to the general scalar range.

208. **Theorem.** If the characteristic equation of a commutative algebra whose coordinates are unrestricted has no multiple roots it is reducible to the sum of r algebras each of one unit, its partial modulus. Such algebra is a WEIERSTRASS algebra.[2]

209. **Theorem.** If the characteristic equation of a commutative algebra has p distinct multiple roots, it is reducible to the sum of p commutative Peirce algebras. Such algebra is a KRONECKER algebra.[3]

210. **Theorem.** The basis of a commutative Peirce algebra is a commutative algebra.

211. **Theorem.** A KRONECKER algebra may contain nilpotents, a WEIERSTRASS algebra can not contain nilpotents.[4] A WEIERSTRASS algebra has nilfactorials.

212. **Theorem.** If the coefficients are restricted to a range, such as a field or a domain of rationality, the algebra may not contain either nilfactorials or nilpotents. Such cases occur in the algebras built from Abelian groups. This case leads to the general theorem: If the equation of the algebra is reducible in the given coordinate range, into p irreducible factors, the algebra is reducible to the sum of p algebras and there are nilfactors. Each irreducible factor belongs to one sub-algebra. If an algebra has an irreducible equation in ζ, the general number, such that the resolvent of this equation and its first derivative as to ζ does not vanish, then all its numbers may be brought to the form

$$\zeta = b_0 e_0 + b_1 i + b_2 i^2 + b_3 i^3 + \dots + b_{r-1} i^{r-1}$$

where i is a certain unit of the algebra, and $b_0 \dots b_{r-1}$ belong to the range. If the resolvent vanishes for either a reducible or an irreducible equation, there are nilpotent numbers in the algebra.[5]

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[1] References for certain commutative algebras follow in the next article. On the general problem see STUDY 2; FROBENIUS 2; KRONECKER 1; SHAW 4.

[2] See references for § 215, also KRONECKER 1.

[3] MOORE 1. [4] KRONECKER 1. [5] MOORE 1; KRONECKER 1.

213. Theorem. In canonical form the adjoined unit is of form

$$j = \sum_{s=1}^{p} \lambda_{ss1} + \sum_{s=2}^{p} \lambda_{ss2} + \ldots .$$

There are as many terms of a given weight k as there are basal units with subscripts that appear in terms of weight k.

214. Theorem. The units of a commutative Peirce algebra may be taken of the form.

$$\xi_1^{t_1} \xi_2^{t_2} \ldots \xi_m^{t_m}$$

where $t_i = 0 \ldots \mu_i$; and where $\xi_i^{\mu_i+1}$, for $i < m$, is linearly expressible in terms of higher order.

2. WEIERSTRASS ALGEBRAS.

215. Definition. A WEIERSTRASS algebra is a commutative algebra satisfying the conditions $\zeta_i \zeta_j = \zeta_j \zeta_i$ and whose degree equals its order,[1] and whose coordinates are *real.*

216. Theorem. Numbers whose coefficient $m_r = 0$ are *nilfactorial* ("divisor of zero"). The product of a nilfactor and any number is a nilfactor. There are no nilpotents in the algebra.[2]

217. Theorem. There is at least one number g, such that $e_0, g, g^2 \ldots g^{r-1}$ are linearly independent. The latent equation resulting may be factored into r linear factors, the imaginary factors occurring in conjugate pairs.

218. Theorem. A WEIERSTRASS algebra is reducible to the sum of r' algebras of the form

$$x_i \qquad x_i^2 = x_i \qquad x_i x_j = 0 \qquad i, j = 1 \ldots r' \qquad r = r' + r'' \qquad i \neq j$$

and whose coordinates are scalars, which appear in conjugate forms if imaginary (r'' is the number of algebras admitting imaginaries). Hence the algebras may be taken to be of the form

$$x_i + x_{i+1} \qquad (x_i - x_{i+1}) \sqrt{-1}$$

with real coefficients; or finally we may take the r' algebras as r' independent ordinary complex algebras.

219. Theorem. Nilfactors are numbers belonging to part only of the partial algebras. If $\zeta_{1,2\ldots n}$ has coordinates in the first n algebras but not in the other $r' - n$, $\zeta_{n+1\ldots r'}$ has coordinates only in the algebras from the $n+1$-th to the r'-th, then

$$\zeta_{1\ldots n} \zeta_{n+1\ldots r'} = 0$$

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[1] WEIERSTRASS 2; SCHWARZ 1; DEDEKIND 1, 2; BERLOTY 1; HÖLDER 1; PETERSON 2; HILBERT 1; STOLZ 1; CHAPMAN 3. The sections below are referred to Berloty.

[2] The presence of nilpotents would lower the degree.

IX. ALGEBRAS WITH COEFFICIENTS IN ARBITRARY FIELDS. REAL ALGEBRAS. DICKSON ALGEBRAS.

220. Definition. An algebra is said to belong to a certain field or domain of rationality, when its coordinates are restricted to that field or domain. In particular an algebra is *real*, when its coordinates are real numbers.[1] The term "finite" algebra is used also to mean algebras whose coordinates are in an abstract (Galois) field.

221. Theorem. The coefficients of the characteristic and the latent equations of an algebra are rational functions of the coordinates in the domain $\Omega_{(x,\gamma)}$, which is the domain of the coordinates and the constants[2] γ.

222. Theorem. If new units are introduced by a transformation T rational in Ω_x, the new units are rational in $\Omega_{x'}$; the hypercomplex domain $\Omega_{(x,e)}$ is then identical with the hypercomplex domain $\Omega_{(x',e')}$. Further, if Ω_x contains Ω_γ, it also contains $\Omega_{\gamma'}$.

223. Theorem. If $S.\zeta$ is defined for any domain, then $S.\zeta$ is invariant under any transformation of the units of the algebra and is rational in $\Omega_{x,\gamma}$.

224. Theorem. In any domain there is an idempotent number or all numbers are nilpotent.

225. Theorem. In a Peirce algebra every number $\zeta = \zeta_0 + \zeta_1$, where ζ_0 is a multiple of the modulus, and ζ_1 is a nilpotent rational in $\Omega_{x,\gamma}$. This separation is possible in only one way. We may choose by a rational transformation new units such that

$$e_0'^2 = e_0' \qquad e_i'^{m_i} = 0 \qquad i = 1 \ldots . r - 1$$

The characteristic equation of ζ is $F.\zeta = \zeta^\nu\,[F_1\zeta]^{\nu'_1}$, where $F.\zeta$ is rationally irreducible in $\Omega_{x\gamma}$.

226. Theorem. In any Scheffers algebra, we may choose by transformations rational in $\Omega_{x\gamma}$, the units η which are nilpotent such that

$$\eta_i \eta_j = \Sigma\gamma_{ijk}\eta_k \qquad k > i,\ k > j$$

227. Definition. A real algebra may be in one of two classes, the real algebras of the first class are such that their characteristic equations have no imaginary roots for any value of ζ, the general number; the second class are such that their characteristic equations in ζ have pairs of conjugate imaginaries.[3]

228. Theorem. Every real quadrate is, if in the first class, of the form given by

$$(1) \quad e_{ij}e_{kl} = \vartheta_{jk}e_{il} \qquad i = 1 \ldots . p$$

If of the second class, it is of order $4p^2$, and is the product of Q and an algebra of the first class (1).

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[1] DICKSON 5; TABER 4. Hamilton restricted Quaternions to real quaternions, calling quaternions with complex coordinates, biquaternions.

[2] TABER 4. The succeeding sections are referred to Taber 4. This paper contains other theorems.

[3] CARTAN 2. This reference applies to §§228–232.

The algebra Q is Quaternions in the Hamiltonian form

$$e_0,\ i,\ j,\ k, \qquad ij = -ji = k, \text{ etc.}$$

229. Theorem. Every real Dedekind algebra is the sum of algebras, each of which is of one of the following three types:

(1) Real quadrates of first class;

(2) Real quadrates of second class;

(3) The product of a quadrate of first class and the algebra e_0, e_1, where

$$e_0^2 = e_0 \qquad e_0 e_1 = e_1 e_0 = e_1 \qquad e_1^2 = -e_0$$

230. Theorem. Every real Scheffers algebra of the second class is derivable from one of the first class by considering that each partial modulus belonging to a complex root of the characteristic equation will furnish two units for the derived algebra, say

$$e_1 = x_1 + x_2 \qquad e_2 = (x_1 - x_2)\sqrt{-1}$$

That is, the direct sub-algebra consists of direct nil-potent units and of the sum of algebras of the forms

$$e_0 \text{ or } e_0, e_1 \qquad (e_1^2 = -e_0)$$

All other units are chosen to correspond; thus η_{2a} furnishes two units, η_{2a}' and η_{2a}'', corresponding to x_2, $\sqrt{-1}\,x_2$.

231. Theorem. A Cartan real algebra is *primary*, and has a Dedekind sub-algebra according to §229, the other units conforming to this sub-algebra in character, and giving multiplication constants γ which are real; or it is *secondary*, and has a Dedekind sub-algebra consisting of the algebras in §229 multiplied by real quadrates of the first class, the other units conforming as usual.

232. Theorem. Every real irreducible (in realm of real numbers) commutative algebra is of the types of §230. It is a Peirce algebra then, the modulus being irreducible; or else it has two partial moduli which give an elementary Weierstrass algebra, and hence are irreducible in the domain of real numbers.

233. Theorem. The only real algebras in which division is unambiguous division (in domain of reals) are (1) real numbers; (2) the algebra of complex numbers $e_0 = e_0^2 = -e_1^2 \quad e_1 = e_0 e_1 = e_1 e_0$ (3) real quaternions.[1]

234. Definition. A. DICKSON algebra is one whose coordinates are in an abstract field.

235. Theorem. The only DICKSON algebras (associative) which admit of division are those whose coordinates are in the Galois (abstract) field, and whose qualitative units are real quaternions or sub-algebras of quaternions.[2]

[1] FROBENIUS 1; C. S. PEIRCE 4; CARTAN 2; GRISSEMANN 1.

[2] WEDDERBURN 4. See also Dickson 6 and 7.

X. NUMBER THEORY OF ALGEBRAS.

236. Definition. The number theory of an algebra is the theory of domains of numbers belonging to that algebra. Algebras usually do not admit of division, unambiguously, hence the term domain is taken here to mean an *ensemble* of numbers such that the addition, subtraction, or multiplication of any of the *ensemble* give a result belonging to the ensemble. The first case which has been studied is that of quaternions, which admits division.[1]

237. Definitions. An infinite system of quaternions is a *corpus* if in this system addition, subtraction, multiplication, and division (except by 0) are determinate uniquely.

A *permutation* of the *corpus* is given by $\left\{ {a \atop f(a)} \right\}$ if through the application of this substitution, every equation between quaternions in the corpus remains an equation. Hence

$$f(a+b) = f(a) + f(b) \qquad\qquad f(ab) = f(a) \cdot f(b)$$

If Ω is the *corpus* of all quaternions we have the substitutions

(1) $q\,(\;)\,q^{-1}$
(2) $f(a) = a_0 \pm a_1 i_\alpha \pm a_2 i_\beta \pm a_3 i_\gamma$ α, β, γ is a permutation of the indices 1, 2, 3.

238. Theorem. If R is the *corpus* of *rational* quaternions, then a is rational when a_0, a_1, a_2, a_3 are rational. The permutations for the rational corpus are $q\,(\;)\,q^{-1}$, and (α, β, γ).

239. Definitions. If $\rho = \frac{1}{2}(1 + i_1 + i_2 + i_3)$, and $q = k_0\rho + k_1 i_1 + k_2 i_2 + k_3 i_3$, where k_0, k_1, k_2, k_3 are any integers, q is said to belong to the integral *domain J.*

If the coordinates of q, $\frac{1}{2}k_0$, $\frac{1}{2}k_0 + k_1$, $\frac{1}{2}k_0 + k_2$, $\frac{1}{2}k_0 + k_3$ are integers, q belongs to the sub-domain J_0.

An *integral* quaternion is one which belongs to J.

An integral quaternion a is *pre-divisible* (*post-divisible*) by b if $a = bc$ ($a = cb$) for some integral quaternion c.

If ε and ε^{-1} are both integral, ε is a *unit.* It follows that

$$N(\varepsilon) = (T\varepsilon)^2 = 1$$

There are 24 units:

$$\varepsilon = \pm 1,\ \pm i_1,\ \pm i_2,\ \pm i_3,\ \frac{\pm 1 \pm i_1 \pm i_2 \pm i}{2}$$

240. Theorem. If $a = vc$, then $a = c_1 v$ only if $v = r\varepsilon$ or $r\zeta\varepsilon$, where $\zeta = 1 + i_1$, and r is any real integer.

[1] HURWITZ 1. Cf. LIPSCHITZ 2. The first reference applies to all sections following to § 257.

241. Theorem. If a and b are integral quaternions, $b \neq 0$, we can find q, c, q_1, c_1 so that

$$a = qb + c \qquad a = bq_1 + c_1 \qquad N(c) < N(b) \qquad N(c_1) < N(b)$$

242. Theorem. Every two integral quaternions a and b, which are not both zero, have a highest common post-factor of the form

$$d = ga + hb \qquad (g, h, \text{ integers})$$

and a highest common pre-factor of the form

$$d_1 = ag_1 + bh_1 \qquad (g_1, h_1, \text{ integers})$$

243. Theorem. The quaternions 0, 1, ρ, ρ^2 form a complete system of residues modulo ζ.

244. Theorem. A quaternion belongs to J_0 if it is congruent to zero or 1 (mod ζ).

245. Theorem. If $N \,.\, a \,(\equiv N \,.\, K a)$ is divisible by 2^r, then $a = \zeta^r b$ where b has an odd norm.

246. Theorem. The following quaternions form a complete system of residues, modulo 2:

$$1,\ i_1,\ i_2,\ i_3 \qquad \frac{1 \pm i_1 \pm i_2 \pm i_3}{2} \qquad 0,\ 1 + i_1,\ 1 + i_2,\ 1 + i_3$$

247. Definition. A *primary* quaternion is one which is congruent to either 1 or $1 + 2\rho$ (mod 2ζ). Every primary quaternion belongs to J_0.

Two integral quaternions are *pre-* (*post-*) *associated*, if they differ only by a pre- (post-) factor which is a unit.

248. Theorem. Of the 24 quaternions associated to an odd quaternion b, only one is *primary*.

249. Theorem. The product of two primary quaternions is primary.

250. Theorem. If b is primary then when

(1) $b \equiv 1 \pmod{2\zeta}$, $K \,.\, b$ is primary,
(2) $b \equiv 1 + 2\rho \pmod{2\zeta}$, $-K \,.\, b$ is primary.

251. Theorem. If m is a positive odd number, the m^4 quaternions

$$q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 \qquad (q_0, q_1, q_2, q_3 = 0, 1, 2 \ldots . m - 1)$$

form a complete system of residues modulo m.

These quaternions are holoedrically isomorphic with the linear homogeneous integral binary substitutions:

$$x_1' \equiv \alpha x_1 + \beta x_2 \qquad x_2' \equiv \gamma x_1 + \delta x_2 \qquad \pmod{m}$$
$$N(\alpha\delta - \beta\gamma) \equiv N \,.\, q \qquad \pmod{m}$$

252. Theorem. The number of solutions of $N(q) \equiv 0 \pmod{m}$, q being prime to m, is $m^3 \Pi \left(1 - \frac{1}{p^2}\right)\left(1 + \frac{1}{p}\right)$

The number of solutions of $N(q) \equiv 1 \pmod{m}$ is $m^3 \Pi \left(1 - \frac{1}{p^2}\right)$. These form a group G_m which is holoedrically isomorphic to the group of the linear homogeneous binary unimodular integral substitutions, modulo m.

253. Definition. π is a *prime* quaternion when its norm is prime.

254. Theorem. There are $p + 1$ primary prime quaternions whose norm equals the odd prime p.

255. Theorem. If $N.c = p^h q^k \dots$ then $c = \pi_1 \pi_2 \dots \pi_h \varkappa_1 \dots \varkappa_k \dots$ where $\pi_i, \varkappa_i \dots$ are primary prime quaternions of norms p, q, etc.

256. Theorem. If m is any odd number, there are $\phi(m) = \Sigma . d$ (sum of the divisors of m) primary quaternions whose norm equals m.

257. Theorem. The integral substitutions of positive determinant which transform $x_0^2 + x_1^2 + x_2^2 + x_3^2$ into a multiple of itself are given by the equations

$$y = \alpha x \beta \qquad y = -\alpha . Kx . \beta$$

where α, β are any two integral quaternions which satisfy the conditions $\alpha\beta \equiv 0$ or $1 \pmod{\zeta}$.

258. Definition. The general number theory of quadrates has been studied recently by DU PASQUIER.[1] A number in a quadrate algebra he calls a *tettarion.* It is practically a (square) matrix or a linear homogeneous substitution. An infinite system of tettarions is a *corpus,* if when α and β belong to the system, $\alpha \pm \beta$, $\alpha . \beta$, $\beta . \alpha$, $\alpha : \beta$, $\beta^{-1} . \alpha$ belong equally to the system. A *substitution* of a tettarion $\bar{\tau} = f(\tau)$ for a tettarion τ is indicated by $[\tau, f(\tau)]$. A *permutation* is a substitution such that when α is derived from n tettarions $\alpha_1 \dots \alpha_n$ by any set of rational operations, so that $\alpha = R(\alpha_1 \dots \alpha_n)$, then $f(\alpha) = \bar{\alpha}$ is derived from $\bar{\alpha}_1 \dots \bar{\alpha}_n$ by the same set of rational operations, so that

$$\bar{\alpha} = R(\bar{\alpha}_1 \dots \bar{\alpha}_n)$$

259. Theorem. The substitution $[\alpha, f(\alpha)]$ is a *permutation* of the corpus $\{K\}$, if the tettarions $f(\alpha)$ do not all vanish, and if

$$f(\alpha + \beta) = f(\alpha) + f(\beta) \qquad f(\alpha\beta) = f(\alpha) f(\beta)$$

for any two tettarions α, β in $\{K\}$. The tettarions $f(\alpha)$ also constitute a corpus.

260. Definition. An *inversion* of the corpus is a substitution such that not all $f(\tau)$ are zero, and also for any two tettarions α, β we have

$$f(\alpha + \beta) = f(\alpha) + f(\beta) \qquad f(\alpha\beta) = f(\beta) f(\alpha)$$

$[\tau, \breve{\tau}]$ is an inversion, where $\breve{\tau}$ is the transpose of τ. If $[\alpha, f(\alpha)]$ is the most general substitution of the corpus, $[\alpha, f(\breve{\alpha})]$ is the most general inversion.

261. Definition. Two permutations of the form

$$[\alpha, f(\alpha)] \text{ and } [\alpha, q . f(\alpha) . q^{-1}]$$

where q is any tettarion which has no zero-roots, are said to be *equivalent.* All equivalent permutations constitute a *class.*

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[1] DU PASQUIER 1. This reference applies to §§ 258–297.

262. Theorem. The substitution $t\alpha t^{-1}$ is a permutation of the corpus Ω of all tettarions of order s; where t is a tettarion such that $N(t) \neq 0$, $N(t)$ being the s-th or last scalar coefficient in the characteristic equation of t. The coefficient $N(t)$ is called the *norm* of t.

263. Definitions. A tettarion is *rational* if all its s^2 coordinates are rational. All rational tettarions form a corpus $\{R\}$. All tettarions whose coordinates belong to a given domain of rationality constitute likewise a corpus. A rational tettarion is integral if all its coordinates are rational integers.

The integral tettarion α is pre- (or post-) divisible by the tettarion β if an integral tettarion γ can be found such that $\alpha = \beta\gamma$ (or $\alpha = \gamma\beta$). A unit tettarion ε is an integral tettarion which is pre- (post-) divisible into every integral tettarion. When $N(\varepsilon) = +1$ we call ε a proper unit-tettarion; when $N(\varepsilon) = -1$ we call ε an improper unit-tettarion.

264. Theorem. Let $\alpha_{ij} = h + e_{ij}$, where h is the modulus of the quadrate, that is, is scalar unity, and e_{ij} is one of the s^2 units defining the quadrate; and let $\tau = \sum_{i,j}^{1\ldots s} t_{ij} e_{ij}$ be any integral tettarion; then among the tettarions

$$\tau^{(x)} = \alpha_{ij}^{x} \tau \qquad (x = 1, 2 \ldots .)$$

there is always one such that a certain pre-assigned coordinate, say $t_{ij}^{(x)}$, is not negative, and is less than the absolute magnitude of any other coordinate of τ of the form t_{kj} $(k = 1 \ldots . s,\ k \neq i)$, provided $t_{kj} \neq 0$.

265. Definition. A tettarion $\Sigma\, t_{ij} e_{ij}$ in which all coordinates for which $i > k$ (or $i < k$) vanish, is said to be *pre-* (*post-*) *reduced.* They constitute a sub-corpus. Tettarions of the form $\Sigma\, t_{ii} e_{ii}$ are both pre-reduced and post-reduced. The components t_{ii} $(i = 1 \ldots . s)$ in a reduced tettarion τ vanish only when τ has zero-roots.

266. Theorem. If τ is any integral tettarion, a proper unit-tettarion ε may be found such that $\varepsilon \cdot \tau$ (or $\tau \cdot \varepsilon$) is a pre- (post-) reduced tettarion, in which, of all the coordinates t_{ii}, at most only t_{ss} can be negative. This coordinate is negative only if $N(\tau) < 0$.

If τ is any integral tettarion, we may find a pair of proper unit-tettarions ε_1 and ε_2 such that $\varepsilon_1 \tau \varepsilon_2$ is of the form $\Sigma\, d_{ii} e_{ii}$ $(i = 1 \ldots . s)$, and among the coordinates at most only d_{ss} is negative, and d_{ii} is divisible by $d_{i-1\, i-1}$. The coordinate d_{ss} is negative only when $N(\tau)$ is negative.[1]

If $\alpha = \varepsilon_1 \beta \varepsilon_2$, α and β are said to be *equivalent.*

267. Theorem. Every proper unit-tettarion ε is expressible in an infinite number of ways as the product of integral powers of at most three unit-tettarions. These three may be

$$\begin{aligned} \alpha_{ij} &= h + e_{ij} \\ \beta_{ij} &= \Sigma\, e_{kk} + e_{ij} - e_{ji} \qquad (k = 1 \ldots . s,\ k \neq i,\ k \neq j) \\ \gamma &= e_{21} + e_{32} + \ldots . + e_{s\,s-1} - e_{1s} \end{aligned}$$

[1] Cf. Kronecker: *Crelle* 107, 135–136; Bachman: *Zahlentheorie* IV Teil, 294.

268. Theorem. Every integral tettarion τ is equivalent to a tettarion of the form $\Sigma\, t_{ii}\, e_{ii}$. The coordinates less τ are the shear factors of the characteristic equation of τ. The norm of τ, $N(\tau)$, is the product of these coordinates. Two tettarions are equivalent when they have the same shear factors and the same nullity.

269. Theorem. In order that α_i be a factor of $\tau = \alpha_1 \ldots\ldots \alpha_i \ldots\ldots \alpha_s$ it is necessary and sufficient that the nullity of α_i be not higher than that of τ, and that each shear factor of α_i, or combination of shear factors, be divisible into the corresponding shear factors of τ. If an integral tettarion τ is a product of others, then every combination of shear factors of τ is divisible by the corresponding combination of shear factors of any one of these others.

270. Definition. Two tettarions τ and $\varepsilon\tau$ are called *pre-associated*. The association is *proper* or *improper* according as $N(\varepsilon) = +1$ or -1. Associated tettarions form a *class*. The simplest representative of a class will be called a *primary* tettarion.

A pre-primary tettarion $\rho = \Sigma p_{ij}\, e_{ij}$ satisfies the following conditions:

$$p_{ij} = 0 \quad i > j \qquad p_{jj} \geqq 0 \text{ and } 0 \leqq p_{ij} < p_{jj}$$

for all $i < j$ and if $p_{jj} \neq 0$; i and j have values $1 \ldots\ldots s$ subject to the conditions stated.

271. Theorem. A primary tettarion cannot have a negative norm. Primary tettarions with zero-roots are infinite in number, but the number of primary tettarions of a given norm $m \neq 0$ is finite.

272. Theorem. If $m = \prod\limits_{i}^{1\ldots n} p_i^{a^{(i)}}$, where p_i is a prime number, and if $\chi(m)$ is the number of distinct pre-primary tettarions of norm m, then

$$\chi(m) = \chi(p_1^{a^{(1)}})\, \chi(p_2^{a^{(2)}}) \ldots\ldots \chi(p_n^{a^{(n)}})$$

$$\chi(p^a) = \frac{(p^{a+1} - 1)\,(p^{a+2} - 1) \ldots\ldots (p^{a+s-1} - 1)}{(p - 1)\,(p^2 - 1) \ldots\ldots (p^{s-1} - 1)}$$

273. Theorem. If τ is any integral tettarion and m is any integer $(\neq 0)$ then an integral tettarion σ may be found such that $\tau = m\sigma$ or else $\tau = m\sigma + \alpha$ wherein α is an integral tettarion and $0 < |N(\alpha)| < |m^s|$.

274. Theorem. If α and δ are two integral tettarions of which δ has not zero-roots, then two integral tettarions β and γ may always be determined such that either

$$\alpha = \beta\delta \text{ and } \gamma = 0 \text{ or } \alpha = \beta\delta + \gamma \text{ and } 0 < |N(\gamma)| < |N(\delta)|$$

By this theorem a highest common pre- (post-) divisor may be found by the Euclidean method for any two integral tettarions.

275. Definition. An infinite system of tettarions which do not all vanish is a *pre-* (*post-*) *ideal* if when τ_i and τ_j belong to the system, $\gamma\tau_i$, $\tau_i \pm \tau_j$ also belong to the system, where γ is any integral tettarion.

276. Theorem. If $\tau_1 \ldots \tau_n$ are integral tettarions, which do not all vanish, then the totality of tettarions $\gamma_1 \tau_1 + \ldots \gamma_n \tau_n$ where $\gamma_1 \ldots \gamma_n$ run independently through the range of all integral tettarions, is a *post-ideal.* The tettarions $\tau_1 \ldots \tau_n$ are said to form the *basis.* An ideal with one tettarion in its basis is a *principal ideal.* It is designated by $(\tau\gamma)$ or $(\gamma\tau)$. Two ideals are *equal* if they contain the same tettarions.

277. Theorem. Pre- (post-) associated tettarions generate the same principal post- (pre-) ideal. If two integral tettarions without zero-roots generate the same principal post- (pre-) ideal they are pre- (post-) associated.

278. Theorem. Every pre- (post-) ideal generated by rational integral tettarions which do not all have zero-roots is a principal ideal.

279. Definition. *Nul-ideals* contain only tettarions with zero-roots.

280. Theorem. An ideal which is both pre-ideal and post-ideal cannot be a nul-ideal.

281. Theorem. Every n given integral tettarions $\alpha, \beta \ldots \mu$ which do not all have zero-roots possess a highest common divisor δ which is expressible in the form

$$\delta = \alpha\delta_1 + \beta\delta_2 + \ldots + \mu\delta_n \quad \text{or} \quad \delta = \delta_1\alpha + \delta_2\beta + \ldots + \delta_n\mu$$

wherein $\delta_i \, (i = 1 \ldots n)$ are definite integral tettarions. Every pre- (post-) divisor of δ is a common divisor of $\alpha \ldots \mu$ and conversely. Moreover δ is determined to a factor which is a post- (pre-) unit-tettarion.

282. Theorem. If α and β have no common pre- (post-) factor, then two tettarions γ, θ may be found such that

$$\alpha\gamma + \beta\theta = 1 \quad \text{or} \quad \gamma\alpha + \theta\beta = 1$$

If α and β have a common factor these equations cannot be solved. If α and β have a common divisor which is not a unit-tettarion then $N(\alpha)$ and $N(\beta)$ have a common divisor other than unity. The converse of this theorem and the theorem imply each other if one of the tettarions is real, that is, of the form $a \sum_{i=1}^{s} e_{ii}$.

283. Definition. An integral tettarion which is not a unit-tettarion nor has zero-roots is *prime* if it cannot be expressed as the product of two integral tettarions neither of which is a unit tettarion.

284. Theorem. The necessary and sufficient condition that π is a prime tettarion is that its norm $N(\pi)$ is a rational prime number. There are $\dfrac{p^s - 1}{p - 1}$ different primary primes of norm p.

285. Theorem. If δ is an integral tettarion of the form $\Sigma d_{ii} e_{ii}$ and if $N(\delta) = p^a q^b \ldots t^r$, where $p, q \ldots t$ are distinct primes, not including unity, then δ can be factored into the form

$$\delta = \pi_1 \ldots \pi_a \varkappa_1 \ldots \varkappa_b \ldots \tau_1 \ldots \tau_n$$

where $\pi_1 \ldots . \pi_a$ are primary prime tettarions of norm p, $\varkappa_1 \ldots . \varkappa_b$ are primary prime tettarions of norm $q, \ldots . \tau_1 \ldots . \tau_n$ are primary prime tettarions of norm t, all being of the form

$$\Sigma\, p_{ii} e_{ii} \ldots . \Sigma\, t_{ii} e_{ii}$$

286. Definition. An integral tettarion is *primitive* if its coordinates have no common divisor other than unity. *It is primitive to an integer* m when its coordinates are all prime to m.

Every primary tettarion is also primitive.

287. Theorem. Let γ be a primitive integral tettarion and

$$N(\gamma) = p^{a_1} q^{a_2} \ldots . t^{a_n}$$

where $p, q \ldots . t$ are the prime factors of $N(\gamma)$. Then γ can be decomposed in only one way into the form

$$\gamma = \pi_1 \ldots . \pi_{a_1} \varkappa_1 \ldots . \varkappa_{a_2} \ldots . \tau_1 \ldots . \tau_{a_n} \varepsilon$$

where ε is a unit tettarion, and $\pi_1 \ldots . \pi_{a_1}$ are prime tettarions of norm p, $\varkappa_1 \ldots . \varkappa_{a_2}$ are prime tettarions of norm $q, \ldots . \tau_1 \ldots . \tau_{a_n}$ are prime tettarions of norm t. The product of each l successive factors is primary, where

$$l = 1 \ldots . a_i \quad (i = 1 \ldots . n)$$

288. Definition. If $\alpha_1, \alpha_2 \ldots . \alpha_s$ are s integral tettarions of equal norms $N(\alpha_1) = N(\alpha_2) \ldots . = N(\alpha_s)$, and if $\alpha_1 \alpha_2 \ldots . \alpha_s = N(\alpha_1)$, then these tettarions are *semi-conjugate.*

289. Theorem. A product of any number of prime tettarions of forms $\Sigma\, d_{ii} e_{ii}$ is a primitive tettarion of the same form if among the factors no s of them are semi-conjugate.

290. Theorem. A product of primary prime tettarions $\pi_1 \ldots . \pi_n$, where $N(\pi_i) = p_i \; (i = 1 \ldots . n)$ p_i being distinct primes, is always a primitive tettarion.

291. Definition. Two given tettarions α and β are *pre-* (*post-*) *congruent* to a modulus γ, if their difference $\alpha - \beta$ is pre- (post-) divisible by γ. This congruence is indicated by

$$\alpha \equiv \beta \qquad (\text{mod } \gamma, \text{ pre})$$

or

$$\alpha \equiv \beta \qquad (\text{mod } \gamma, \text{ post})$$

There is then an integral tettarion ζ such that

$$\alpha - \beta \equiv \gamma\zeta \text{ or } \zeta\gamma$$

292. Theorem. If α and β are pre- (post-) congruent modulo γ, they are also pre- (post-) congruent for any tettarion post- (pre-) associated with γ, as modulus.

293. Theorems. If $\alpha \equiv \tau \quad \beta \equiv \tau$ then $\alpha \equiv \beta \pmod{\gamma}$
If $\alpha \equiv \tau \quad \beta \equiv \sigma$ then $\alpha \pm \beta \equiv \tau \pm \sigma \pmod{\gamma}$
If $\alpha \equiv \beta$ then $\tau\alpha \equiv \tau\beta$ (mod γ, post)
If $\alpha \equiv \beta$ then $\alpha\tau \equiv \beta\tau$ (mod γ, pre)
If $\gamma\phi \equiv \psi\gamma \quad \alpha \equiv \beta$ then $\alpha\phi \equiv \beta\psi$ (mod γ, post)
If $\alpha \equiv \beta \quad \theta \equiv \delta \quad \gamma\beta \equiv \zeta\gamma$ then $\theta\alpha \equiv \delta\beta$ (mod γ, post)

294. Definition. Two tettarions α, β are congruent as to a rational integer $m \neq 0$, when $\alpha - \beta$ is divisible by m, indicated by $\alpha \equiv \beta \pmod{m}$. In this case for each pair of coordinates we have $a_{ij} \equiv b_{ij} \pmod{m}$. A complete system of residues consists of m^{s^2} tettarions, obtained by setting each coordinate independently equal to each one of a complete set of residues modulo m.

295. Theorems. A tettarion congruence modulo m, a rational integer can be divided by an integral tettarion ζ, without altering the modulus only if $N(\zeta)$ is prime to m.

A sufficient condition for the solubility of the congruence $\alpha\xi \equiv \beta \pmod{m}$ by an integral tettarion ξ is that $N(\alpha)$ is prime to m. If this condition is fulfilled the congruence possesses one and only one solution $\xi \pmod{m}$, namely, $\xi = r\dfrac{N(\alpha)}{\alpha}\beta \pmod{m}$, where r is a root of $r \,.\, N(\alpha) \equiv 1 \pmod{m}$.

296. Definitions. A tettarion with zero roots and nullity $s - r$ is *pseudo-real* if it is of the form $d_{11} \sum_{t=1}^{r} e_{tt}$, $r < s$. A tettarion with zero-roots, of the form $\Sigma\, t_{ij}\, e_{ij}$ where $t_{ij} = 0$ for $j = r + 1 \ldots . s$, is *singular* or *non-singular* according as its rank is less than or equal to r. The product of the first r latent roots of a tettarion of this kind is called its *pseudo-norm* N'. A tettarion of the type $\Sigma\, t_{i1}\, e_{i1}$ is never singular. When a tettarion is singular its pseudo-norm is zero.

297. Theorem. If α and μ are two integral tettarions in which coordinates of the form $t_{ij} = 0$ for $j = r + 1 \ldots . s$, and if α is pre-reduced, μ is pseudo-real, and $\neq 0$, then two tettarions of the same type ζ and η, may always be found such that either

$$\alpha = \mu \,.\, \zeta \qquad \eta = 0$$

or

$$\alpha = \mu\zeta + \alpha$$

where the pseudo-norm of α satisfies the conditions

$$0 < |N'(\alpha)| < |N'(\mu)| = |(m_{11})^r|$$

If τ and β are two integral pre-reduced tettarions of the same type as α, μ above, and β is non-singular, then there are always two other pre-reduced tettarions of this type ζ and η such that

$$\tau = q\beta \qquad \eta = 0$$

or

$$\tau = q\beta + \eta \quad \text{and} \quad 0 < |N'(\eta)| < |N'(\beta)|$$

Every non-singular post-ideal based on tettarions of this type is a principal ideal.

XI. FUNCTION THEORY OF ALGEBRAS.

298. **Definition.** In §58, chapter II, we have for any analytic function of ζ,

$$F.\zeta = \sum_{i=1}^{g} \left\{ F.g_i x_i + F' g_i \Phi_i + \frac{F''}{2!} g_i \Phi_i^2 + \dots + \frac{F^{(\mu_{i1}-1)}}{(\mu_{i1}-1)!} g_i \Phi^{\mu_{i1}-1} \right\}$$

This definition gives a complete theory, if the roots may be treated as known. Other definitions are given below.[1]

299. **Definition.** $\sum_1^\infty a_\nu \zeta^\nu$ defines an analytic function of ζ, if the roots of the characteristic equation of ζ converge in the circle defined by $\sum_1^\infty a_\nu z^\nu$, where z is an ordinary complex number.

$\sum_{-\infty}^{+\infty} a_\gamma \zeta^\nu$ defines a function of ζ, if ζ^{-1} exists, and if the roots of the characteristic equation of ζ converge in the circles[2] of $\sum_1^\infty a_\gamma z^\nu$ and $\sum_1^\infty a_{-\gamma} z^{-\nu}$.

300. **Definition.** Let

$$f = \Sigma e_i f_i (x_1 \dots x_r) \qquad i = 1 \dots r$$

and let

$$dx = \Sigma e_i dx_i$$

Then

$$df = \sum_{i,k}^{1\dots r} \frac{\partial f_i}{\partial x_k} . dx_k e_i = f' . dx = dx . 'f$$

If $\frac{dx_i}{dt} = y_i$, then f is an analytic function[3] when $f'.y = y.'f$.

301. **Theorem.** The algebra must contain for every number u, a number v such that $uy = yv$ for every y.

302. **Theorem.** In a commutative algebra the necessary and sufficient condition of analytic functions is

$$\frac{\partial f_s}{\partial x_k} = \sum_{i,j}^{1\dots r} \gamma_{iks} \varepsilon_j \frac{\partial f_i}{\partial x_j} \qquad s, k = 1 \dots r$$

303. **Theorem.** The derivative of an analytic function is an analytic function. If two analytic functions have the same derivative, they differ only by a constant.

304. **Theorem.** An analytic function is a differential coefficient, only when the algebra is associative, distributive, and commutative.

305. **Theorem.** If u and v are analytic, uv and $\frac{u}{v}$ are analytic.

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[1] FROBENIUS 1; BUCHHEIM 7; SYLVESTER 4; TABER 6, 7. [2] WEYR 7.
[3] SCHEFFERS 8. Applies to §§300–306.

306. Theorem. If $f(\rho x) = f(x)$, then $f(x)$ is a constant.

307. Theorem. For a WEIERSTRASS commutative algebra, let a_i, b_i be in the i-th elementary algebra, and

$$a = \Sigma\, a_i \qquad b = \Sigma\, b_i$$

Then $a \pm b = \Sigma\,(a_i \pm b_i)$ $\quad ab = \Sigma\, a_i b_i \quad \frac{a}{b} = \Sigma\, \frac{a_i}{b_i}$, if b is not a nilfactor.

If $\qquad b_i = 0$, for $i = 1 \ldots . i_1$, $b_i \neq 0 \qquad i > i_1$

and if $\qquad a_i = 0$, for $i = 1 \ldots . i_1$

then $\qquad \frac{a}{b} = \Sigma \frac{a_i}{b_i} \qquad i > i_1$

In any other case the division of a by b gives an *infinity*.[1]

308. Theorem. The sum, difference, product, and quotient of two polynomials is formed as in ordinary algebra.

309. Theorem. The number of solutions of an algebraic equation of degree p is $N = p^{\frac{1}{2}r}$, when each elementary algebra is of order two.

If r_1 of the elementary algebras are of order one, and $r - r_1$ of order two, $N = p^r$.

In any case the number of infinities and roots is p^r. The number of roots is infinite if, and only if, the coefficients are multiples of the same nilfactor.[2]

310. Theorem. A polynomial $F(\zeta)$ can not vanish for every value of ζ unless its coefficients all vanish.

Two polynomials equal to each other for every value of ζ, must have the coefficients of like powers of ζ equal.

311. Theorem. If an algebraic polynomial $F(\zeta)$ is divided by $\zeta - \zeta'$, ζ' being a root, the degree is reduced to $(p-1)$ and $p^{\frac{1}{2}r} - (p-1)^{\frac{1}{2}r}$ roots have been removed.

In ordinary complex algebra $r = 2$, $\quad p^{\frac{1}{2}r} - (p-1)^{\frac{1}{2}r} = 1$.

312. Theorem. If two polynomials have a common root, ζ', they have a common divisor $\zeta - \zeta'$.

313. Theorem. If $F(\zeta)$ is differentiated as if ζ were an ordinary quantity, giving $F'(\zeta)$, then the necessary and sufficient condition that there is a system of roots of $F(\zeta)$, having just p equal roots, is that $F'(\zeta)$ has at least one system of roots of which $p-1$ are this same equal root, and that no system of roots of $F'(\zeta)$ has this root more than $p-1$ times. $F(\zeta)$ and $F'(\zeta)$ have therefore the common divisor $(\zeta - \zeta')^{p-1}$.

314. Theorem. It is not always possible to break up $\frac{f(\zeta)}{F(\zeta)}$ into partial fractions.

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[1] BERLOTY 1. Applies to §§307-315. [2] WEIERSTRASS 2.

315. Theorem. If ζ is considered to be written in the form $\zeta = \Sigma z_i x_i$, where $i = 1 \ldots . r_1$, and z_i is any real or complex number, the whole theory of functions of a complex variable may be extended to numbers which are not nilfactors. If there are nilfactors, meromorphic functions must be treated specially. We have

$$F . \zeta = \Sigma x_i F . z_i$$

316. The treatment of quaternion and biquaternion differentials, integrals, and functions may be found in the treatises on these subjects and references there given; references are also given at the end of this memoir. The general principles of such forms may easily be extended to any algebra. Differentiation and integration along a line, over a surface, etc., may also be found in the appropriate treatises.

The problem of extending monogeneity to functions of numbers in quadrate algebras has been handled recently by AUTONNE.[1] His results are as follows:

Let ξ be any number in an algebra, and let Ξ be a number whose coordinates are functions of those of ξ. The *index of monogeneity* N is the minimum number of terms necessary to write $d\Xi$ in the form $\sum_{i=1}^{N} \sigma_i . d\xi . \tau_i$, wherein σ_i and τ_i are functions of ξ. If we write $\nabla = \sum_{i=1}^{r} e_i \dfrac{\partial}{\partial x_i}$, we have in all cases $d\Xi = I . d\xi \nabla . \Xi = \Upsilon(d\xi)$. The Jacobian of the coordinates of Ξ is then $m_r(\Upsilon)$.

If now we put $\Upsilon = \sum_{kl}^{1 \ldots . r} w_{kl} K_{kl}$, where $K_{kl} = e_k () e_l$, we may find the scalars w_{kl} uniquely if the algebra is a quadrate.[2] For, indicating quadrate units by a double suffix, and writing $n^2 = r$,

$$\Upsilon = \sum_{ijkl}^{1 \ldots . n} w_{ijkl} K_{ijkl} \qquad (K_{ijkl} = e_{ij} () e_{kl})$$

and if we operate on e_{jk} and take $I . e_{il} ()$ over the result,

$$I . e_{il} \Upsilon e_{jk} = w_{ijkl}$$

If we put $\Psi = \sum_{kl}^{1 \ldots . r} w_{kl} . e_k I e_l ()$, or in the case of a quadrate,

$$\Psi = \sum_{ijkl}^{1 \ldots . n} w_{ijkl} . e_{ij} I e_{kl} ()$$

then the *rank*, that is, $n - \nu$, where ν is the *nullity* of Ψ, is the *index of monogeneity*, N. N is invariant for a change of basis.

The transverse of Ψ corresponds to interchanging σ_i and τ_i. For

$$\breve{\Psi} = \sum_{ijkl}^{1 \ldots . n} I e_{il} \Upsilon e_{jk} . e_{kl} I e_{ij} = \sum_{ijkl}^{1 \ldots . n} I e_{kj} \Upsilon e_{li} . e_{ij} I e_{kl}$$

and $I e_{kj} \Upsilon e_{li} = w_{klij}$.

---

[1] AUTONNE 5, 6. [2] HAUSDORFF 1.

Let $P() = \sum_{i=1}^{r} e_i I\zeta(e_i()\zeta)$, the $\zeta\zeta$ forming a ζ-pair. Then $\breve{P} = P$, and we have

$$\begin{aligned} P\Upsilon &= \sum_{kl} w_{kl} \sum e_i I\zeta(e_i e_k () e_l \zeta) \\ &= \sum w_{kl} \sum e_i I e_j () \,.\, I\zeta(e_j e_l e_i e_k \ \zeta) \\ &= \sum w_{kl} \sum e_j I e_i () \,.\, I\zeta(e_i e_l e_j e_k \zeta) \\ \widetilde{P\Upsilon} &= \sum w_{kl} \sum e_i I e_j () \,.\, I\zeta(e_i e_l e_j e_k \zeta) \end{aligned}$$

Hence $\widetilde{P\Upsilon} = \breve{\Upsilon} P = P\Upsilon$ if $\Psi = \breve{\Psi}$, and conversely. Again $P\Upsilon = P\Xi I\nabla$, therefore if $P\Upsilon = \breve{\Upsilon}P$ we have

$$P\Xi I\nabla = \nabla I P\Xi$$

Operating on $d\xi$, we have

$$d \,.\, P\Xi = \nabla I d\xi P\Xi = \nabla I \Xi d P\xi$$

Hence if $\eta = P\xi$, $I\Xi d\eta$ is an exact differential. Thus if Ψ is self-transverse, $I\Xi d\eta$ is an exact differential and conversely.

When $N = 1$, we have Ξ in one of the four following types:

$$\text{I.} \quad \Xi = K\xi\Lambda + M \qquad (K, \Lambda, M, \text{constant})$$

$$\text{II.} \quad \Xi = \sum_{i=1}^{n} Z(\xi e_{i1}) e_{i1} \qquad (Z \text{ arbitrary})$$

$$\text{III.} \quad \Xi = \sum_{i=1}^{n} e_{i1} Z(e_{i1}\xi) \qquad (Z \text{ arbitrary})$$

$$\text{IV.} \quad \Xi = \sum_{ij}^{1\ldots n} e_{ij} \int \eta_i(t)\, p_j(t)\, dt$$

$t = \psi[I.\alpha(e_{11}\xi) \ldots . I.\alpha\ (e_{nn}\xi)]$, and ψ, η_i, p_j are arbitrary scalar functions of t; α is any constant number.

XII. GROUP THEORY OF ALGEBRAS.

317. This part of the subject is practically undeveloped, although certain results in groups are at once transferable to algebras. A considerable body of theorems may thus be got together, especially for the quadrates. For example, the groups of binary linear homogeneous substitutions lead at once to quaternion groups, ternary linear homogeneous substitutions to nonion groups, etc. It is to be hoped that this branch may be soon completed.[1]

318. **Definition.** A group of quaternions is a set of quaternions $q_1 \dots q_n$, such that

$$q_i^{m_i} = 1 \qquad q_i q_j = q_k \qquad i, j = 1 \dots n$$

m_i is a positive integer, and k has any value $1, 2 \dots n$. The quaternions give *real, complex,* or *congruence* groups according as the coordinates are real, complex, or in an abstract field.

319. **Theorem.** To every quaternion $q = w + xi + yj + zk$ corresponds the linear homogeneous substitution

$$\begin{pmatrix} w + z\sqrt{-1} & -y + x\sqrt{-1} \\ y + x\sqrt{-1} & w - z\sqrt{-1} \end{pmatrix}$$

and conversely. The determinant of the substitution is $T^2 q$. To the product of two quaternions q, r, corresponds the product of the substitutions.

320. **Theorem.** To every group of binary linear homogeneous substitutions corresponds a quaternion group, and conversely. To every group of binary linear fractional unimodular substitutions corresponds a group of quaternions multiply isomorphic with it, and to every quaternion group corresponds a group of binary linear fractional unimodular substitutions, the latter not always distinct for different quaternion groups.

321. **Theorem.** To every quaternion of tensor Tq corresponds a Gaussian operator $Tq \,.\, q\,()\,q^{-1} = G_q$, and conversely.

If $q \,.\, r = s$, then $G_q \,.\, G_r = G_s$.

Hence groups of these Gaussian operators are isomorphic with quaternion groups, and conversely, but the isomorphism is not one-to-one.

322. **Theorem.** To every unit quaternion q, there corresponds a rotator $R_q = q\,()\,q^{-1}$, and conversely, the same rotator corresponding to more than one quaternion.

Likewise a reflector $\bar{R}_q = -q\,()\,q^{-1}$, and conversely.

Further, for any fixed quaternion a admitting of a reciprocal, there corresponds the *a-transverse* of q,

$$T_q^{(a)} = a\,\bar{q}\,a^{-1}$$

---

[1] Cf. Laurent 3, 4.

Thus if $qr = s$,

$$R_q \,.\, R_r = R_s \qquad \bar{R}_q \,.\, \bar{R}_r = -\, R_s \qquad T_r^{(a)} \,.\, T_q^{a} = T_s^{(a)}$$

Thus to every group of quaternions $q_1 \ldots . q_n$, corresponds the rotator group $R_{q_1} \ldots . R_{q_n}$; the reflector group $\pm \bar{R}_{q_1}$, $\pm \bar{R}_{q_2} \ldots . \pm \bar{R}_{q_n}$; and the transverse groups $T_{q_1}^{(a)} \ldots . T_{q_n}^{(a)}$. If $a = 1$, the transverse group is the group of conjugates; and if $Sa = 0$, we have a group of transverses in the matrix sense.

323. Theorem. If we consider that q and $-q$ are to be equivalent, $q \equiv -q$, then the rotation groups give the quaternion groups as follows:

To C_n corresponds $k^{\frac{2r}{n}}$, $\qquad r = 1 \ldots . n$

D_n corresponds $k^{\frac{2r}{n}}$, i, $\qquad r = 1 \ldots . n$

T corresponds $1, i, j, k, \ (1 \pm i \pm j \pm k)$

O corresponds $1, i^{\frac{r}{2}}, j^{\frac{r}{2}}, k^{\frac{r}{2}}, \tfrac{1}{2}(1 \pm i \pm j \pm k)$

$\tfrac{1}{2}\sqrt{2}\,(i \pm j), \ \tfrac{1}{2}\sqrt{2}\,(j \pm k), \ \tfrac{1}{2}\sqrt{2}\,(k \pm i) \qquad r = 1, 2, 3$

$$I \text{ corresponds } k^{\frac{2h}{5}}, \ jk^{\frac{2h}{5}}, \ \frac{k^{\frac{2h'}{5}}(i + 2k \cos 72°)\, k^{\frac{2h''}{5}}}{\sqrt{1 + 4\cos^2 72°}}$$

$$\frac{k^{\frac{2h'}{5}}\, j(1 + 2k \cos 72°)\, k^{\frac{2h''}{5}}}{\sqrt{1 + 4\cos^2 72°}} \qquad h, h', h'' = 1 \ldots . 5$$

324. Theorem. To the extended polyhedral groups correspond the following five quaternion groups:

To C'_r corresponds the group $k^{\frac{4n}{r}}$, of order r, (k any unit vector, $n = 1 \ldots . r$).

To D'_r corresponds the group $k^{\frac{4n}{r}}\, i^h$, of order $4r$, ($Sik = 0$, $i^2 = -1$, $n = 1 \ldots . r$; $h = 1 \ldots . 4$).

To T' corresponds the group of order 24: $\pm 1, \pm i, \pm j, \pm k$, $\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$.

To O' corresponds the group of order 48: $\pm 1, \pm i, \pm j, \pm k$,
$\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k) \quad \tfrac{1}{2}\sqrt{2}\,(\pm 1 \pm i) \quad \tfrac{1}{2}\sqrt{2}\,(\pm 1 \pm j)$
$\tfrac{1}{2}\sqrt{2}\,(\pm 1 \pm k) \quad \tfrac{1}{2}\sqrt{2}\,(\pm i \pm j) \quad \tfrac{1}{2}\sqrt{2}\,(\pm j \pm k)$
$\tfrac{1}{2}\sqrt{2}\,(\pm k \pm i)$

To I' corresponds the group of order[1] 120: $\pm k^{\frac{2n}{5}}$, $\pm jk^{\frac{2n}{5}}$,

$$\frac{\pm k^{\frac{2n}{5}}(i + \omega k)\, k^{\frac{2s}{5}}}{\sqrt{1 + \omega^2}} \qquad \frac{\pm k^{\frac{2n}{5}}(i + \omega k)\, jk^{\frac{2n}{5}}}{\sqrt{1 + \omega^2}}$$

where

$$n, s = 1 \ldots . 5 \qquad n + s \equiv \pm 1, \pm 2 \pmod 5 \qquad \omega = 2\cos 72° = \tfrac{1}{2}(-1 + \sqrt{5})$$

[1] Cf. STRINGHAM 3.

325. **Theorem.** Combinations of rotations and reflections give the polyhedral and the crystallographic groups. Thus we have correspondences:

$$\begin{array}{llr}
C_r = k^{\frac{2n}{r}}()k^{-\frac{2n}{r}} & & n=1\ldots.\ r \\
D_r = k^{\frac{2n}{r}}()k^{-\frac{2n}{r}} \quad i()i^{-1} & & n=1\ldots.\ r \\
T \ = 1 \quad k()k^{-1} \quad i()i^{-1} \quad j()j^{-1} & (1\pm i\pm j\pm k)()(1\pm i\pm j\pm k)^{-1} & \\
O \ = 1 \quad i()i^{-1} \quad j()j^{-1} \quad k()k^{-1} & (1\pm i\pm j\pm k)()(1\pm i\pm j\pm k)^{-1} & \\
\quad (1\pm i)()(1\pm i)^{-1} \quad (1\pm j)()(1\pm j)^{-1} & (1\pm k)()(1\pm k)^{-1} & \\
\quad (i\pm j)()(i\pm j)^{-1} \quad (j\pm k)()(j\pm k)^{-1} & (k\pm i)()(k\pm i)^{-1} & \\
I \ = k^{\frac{2}{5}}()k^{-\frac{2}{5}} \quad j()j^{-1} & (i+2\cos 72^\circ k)()(i+2\cos 72^\circ k)^{-1} &
\end{array}$$

and their combinations.

$$\begin{array}{lllr}
C'_r = [-k^{\frac{r+1}{r}}()k^{-\frac{r+1}{r}}]^h & & & h=1\ldots.2r \\
C''_r = k^{\frac{2h}{r}}()k^{-\frac{2h}{r}} & -k()k^{-1} & \text{and combinations} & h=1\ldots.\ r \\
C'''_r = k^{\frac{2h}{r}}()k^{-\frac{2h}{r}} & -i()i^{-1} & \text{and combinations} & h=1\ldots.\ r \\
D'_r = [-k^{\frac{r+1}{r}}()k^{-\frac{(r+1)}{r}}] & i()i^{-1} & \text{and combinations} & h=1\ldots.2r \\
D''_r = k^{\frac{2h}{r}}()k^{-\frac{2h}{r}} & -k()k^{-1} \quad i()i^{-1} & \text{and combinations} & h=1\ldots.\ r \\
D'''_r = k^{\frac{2h}{r}}()k^{-\frac{2h}{r}} & -\alpha()\alpha^{-1} \quad i()i^{-1} \quad \alpha\perp k & \text{and combinations} & h=1\ldots.\ r \\
T' = T & -T & & \\
T'' = T & [-(i-j)()(i-j)^{-1}] & \text{and combinations} & \\
O' = O & -O & & \\
I' = I & -I & &
\end{array}$$

326. **Theorem.** If $S.\varepsilon=\varepsilon$, $\varepsilon^m=1$; then the product of each group in §324 into the cyclic group of ε, gives a group of quaternions.

327. Groups of quaternions whose coordinates are in an abstract field, remain to be investigated.

328. **Theorem.** The continuous groups[1] of quaternions are as follows:

(1) All quaternions.
(2) All unit quaternions.
(3) Quaternions of the form $w+xi+y\vartheta$; $Si\vartheta=0=\vartheta^2$.
(4) Quaternions of the form $w+y\vartheta$; (ϑ may $=j+\sqrt{-1}\,k$).
(5) Quaternions on the same axis, $w+xi$.
(6) Scalars, w.
(7) The quaternions $t^{c+1}\frac{1}{2}(1+\sqrt{-1}\,i)+t^c\frac{1}{2}(1-\sqrt{-1}\,i)+y\vartheta$, t arbitrary.
(8) The quaternions $e^t+te^t\vartheta$.
(9) The quaternions $1+y\vartheta$.
(10) The quaternions $t^{c+1}\frac{1}{2}(1+\sqrt{-1}\,i)+t^c\frac{1}{2}(1-\sqrt{-1}\,i)$.

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[1] Scheffers 7.

XIII. GENERAL THEORY OF ALGEBRA.

329. While this memoir is particularly concerned with associative linear algebra, it is nevertheless necessary, in order to place the subject in its proper perspective, to give a brief account of what is here called, for lack of a better title, the general theory of algebra.

The foundations of mathematics consist of two classes of things—the *elements* out of which are built the structures of mathematics, and the *processes* by which they are built. The primary question for the logician is: What are the primordial elements of mathematics? He proceeds to reduce these to so-called *logical constants:*[1] *implication, relation of a term to its class,* notion of *such that,* notion of *relation,* and such further notions as are involved in formal implication, viz., *propositional function, class, denoting,* and *any or every term.* To the mathematician these elements do not convey much information as to the processes of mathematics. The life of mathematics is the derivation of one thing from others, the transition from data to things that follow according to given processes of transition.

For example, consider the notions 3, 4, 7. We may say that we have here a case of correspondence, namely to the two notions 3, 4 corresponds the notion 7. But by a different kind of correspondence, to 3, 4 corresponds 12; or by other correspondences 81, or $\sqrt[4]{3}$, and so on. Now it is true that in each case here mentioned we have a kind of correspondence, but these kinds of correspondence are different, and herein lies the fact that all correspondences are processes. Equally, if we say that we have cases of relations,—that 3, 4, 7 stand in one relationship; 3, 4, 12 in another, etc.—these relations are different, and the generic term for all of them is *process.* The psychological fact that we may associate ideas together, and call such association, correspondence, or relationship, functionality, or like terms, should not obscure the mathematical fact, which is equally psychological, that we may pass from a set of ideas to a different idea, or set of ideas, —a mental phenomenon which we may call inference, deduction, implication, etc. We therefore shall consider that any definite rule or method of starting from a set of ideas and arriving at another idea or set of ideas is a mathematical process, if any person acquainted with the ideas entering the process and who clearly understands the process, would arrive at the same goal.

Thus, all persons would say that 3 added to 4 gives 7, 3 multiplied by 4 gives 12, etc., wherein the words add, multiply, etc., indicate definite processes.

330. Definition. A mathematical process is defined thus:

I. Let there be a class of entities $\{a\}$.

II. Let there be chosen from this class $n-1$ entities, in order $a_1, a_2 \cdot \cdot \cdot \cdot a_{n-1}$.

III. Let these entities in this order define a method, F, of selecting an entity, a_n, from the set.

Then $F(a_1, a_2 \cdot \cdot \cdot \cdot a_{n-1}, a_n)$ is said to represent a *mathematical process.*

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[1] B. Russell 1, p. 106.

The entities $a_1 \ldots\ldots a_{n-1}$ are called the first, second ..... $(n-1)$-th *facients* of the process. The entity a_n is called the result. Occasionally this process has been called multiplication, $a_1 \ldots\ldots a_{n-1}$ being called factors.

331. The class of entities $\{a\}$ may be finite or transfinite. If transfinite they may be capable of order, and may be ordered, or they may be chaotic. It is not known whether there is any class incapable of being ordered, or not.

The number n may be any number, finite or transfinite, of a CANTOR ordinal series of numbers.

332. **Definition.** Let us suppose, in the process $F(a_1, a_2 \ldots\ldots a_{n-1}, a_n)$, that a_n is known, but a_r $[1 \overline{\gtrless} r \overline{\gtrless} n-1]$ is not known. We may conceive that by some process F_σ, we can find a_r, the order of the known terms being, let us say,

$$(a_{i_1}, a_{i_2} \ldots\ldots a_{i_{n-1}}, a_r)$$

where $i_1, i_2 \ldots\ldots i_{n-1}$ are the subscripts $1, 2 \ldots\ldots r-1, r+1 \ldots\ldots n$ in some order, so that

$$F_\sigma(a_{i_1}, a_{i_2} \ldots\ldots a_{i_{n-1}}, a_r)$$

F_σ is called a *correlative* process, the σ-correlative of F. The process F is *uniform* when, for all correlative processes, a_r is determined uniquely.

333. Theorem. There are for F, $n!$ correlative processes, including F. We may designate these by the substitutions of the symmetric group on n things; so that if we have

$$F(a_1, a_2, a_3 \ldots\ldots a_n)$$

then we also have

$$F_{\sigma^{-1}}(a_{i_1}, a_{i_2} \ldots\ldots a_{i_{n-1}}, a_{i_n})$$

where σ is the substitution

$$\begin{pmatrix} 1, & 2, & 3 \ldots\ldots n \\ i_1, & i_2, & i_3 \ldots\ldots i_n \end{pmatrix}$$

334. Theorem. Evidently the σ_1-correlative of the σ_2-correlative of F is the σ_3-correlative of F, where

$$\sigma_3 = \sigma_1 \sigma_2$$

We write, therefore, $F_{\sigma_2^{-1}\sigma_1^{-1}} = F_{\sigma_3^{-1}} = F_{(\sigma_1\sigma_2)^{-1}}$.

The correlatives thus form a group of order $n!$.

335. **Examples.**

(1) Let a_3 be *tax-payer*, a_1 be *boy*, a_2 *owner of a dog*, then

$F \quad (a_1 a_2 a_3)$: a boy who owns a dog pays taxes.

$F_{(12)} (a_1 a_2 a_3)$: the possession of a dog by the boy requires payment of the tax.

$F_{(13)} (a_1 a_2 a_3)$: the tax on a dog is paid by the boy.

$F_{(23)} (a_1 a_2 a_3)$: the boy pays taxes on the dog he owns.

$F_{(123)} (a_1 a_2 a_3)$: the tax paid by the boy is on a dog.

$F_{(132)} (a_1 a_2 a_3)$: the dog requires that a tax be paid by the boy.

(2) Let a_1, a_2, a_3 be numbers; $F(a_1 a_2 a_3)$ mean a_3 is the a_1 power of a_2.

Then $F_{(12)}\,(a_1\,a_2\,a_3)$ means a_3 is the log a_2 power of the exponential of a_1.
$F_{(13)}\,(a_1\,a_2\,a_3)$ means a_1 is the quotient of log a_2 by log a_3.
$F_{(23)}\,(a_1\,a_2\,a_3)$ means a_2 is the a_1 root of a_3.
$F_{(123)}\,(a_1\,a_2\,a_3)$ means a_2 is the log a_3 power of the exponential of $\frac{1}{a_1}$.
$F_{(132)}\,(a_1\,a_2\,a_3)$ means a_1 is log a_3 on the base a_2.

336. Theorem. The correlatives of F fall into sub-groups corresponding to the sub-groups of the group $G_{n!}$.

337. Definition. It may happen that in a given process, F, we may have simultaneously for all values of $a_1 \ldots . a_{n-1}$

$$F(a_1\,,\,a_2\,\ldots . a_n) \qquad (1)$$

$$F(a_{i_1},\,a_{i_2} \ldots a_n) \qquad (2)$$

Since we must have from (1) $F_{\sigma^{-1}}(a_{i_1},\,a_{i_2} \ldots . a_n)$ we must identify F and $F_{\sigma^{-1}}$, or as we may write it, $F = F_{\sigma^{-1}}$. The correlatives will break up then into $\frac{n}{m}$ groups where m is the order of the substitution σ. We call these cases *limitation-types* of F.

Examples. For $F(a_1\,a_2)$ we have but one case: $F = F_{(12)}$.

For $F(a_1\,a_2\,a_3)$ we have five types:

(1) $F = F_{(12)}$. This is the familiar commutativity of ordinary algebra. It follows that

$$F_{(13)} = F_{(23)} \qquad F_{(123)} = F_{(132)}$$

(2) $F = F_{(13)}$, whence $F_{(12)} = F_{(132)}$, $F_{(23)} = F_{(123)}$

(3) $F = F_{(23)}$, whence $F_{(12)} = F_{(123)}$, $F_{(13)} = F_{(132)}$

(4) $F = F_{(123)} = F_{(132)}$, whence $F_{(12)} = F_{(13)} = F_{(23)}$

(5) $F = F_{(12)} = F_{(13)} = F_{(23)} = F_{(123)} = F_{(132)}$

For $F(a_1\,a_2\,a_3\,a_4)$ we have twenty-nine types corresponding to the sub-groups of the group $G_{4!}$:

(1) $F = F_{(12)}$ (2) $F = F_{(13)}$ (3) $F = F_{(14)}$
(4) $F = F_{(23)}$ (5) $F = F_{(24)}$ (6) $F = F_{(34)}$
(7) $F = F_{(12)\,(34)}$ (8) $F = F_{(13)\,(24)}$ (9) $F = F_{(14)\,(23)}$
(10) $F = F_{(123)} = F_{(132)}$ (11) $F = F_{(124)} = F_{(142)}$ (12) $F = F_{(134)} = F_{(143)}$
(13) $F = F_{(234)} = F_{(243)}$ (14) $F = F_{(1234)} = F_{(13)\,(24)} = F_{(1432)}$
(15) $F = F_{(1324)} = F_{(12)\,(34)} = F_{(1423)}$ (16) $F = F_{(1342)} = F_{(14)\,(23)} = F_{(1243)}$
(17) $F = F_{(12)} = F_{(34)} = F_{(12)\,(34)}$ (18) $F = F_{(13)} = F_{(24)} = F_{(13)\,(24)}$
(19) $F = F_{(14)} = F_{(23)} = F_{(14)\,(23)}$ (20) $F = F_{(12)\,(34)} = F_{(13)\,(24)} = F_{(14)\,(23)}$
(21) $F = F_{(12)} = F_{(13)} = F_{(23)} = F_{(123)} = F_{(132)}$
(22) $F = F_{(12)} = F_{(14)} = F_{(24)} = F_{(124)} = F_{(142)}$
(23) $F = F_{(13)} = F_{(14)} = F_{(34)} = F_{(134)} = F_{(143)}$
(24) $F = F_{(23)} = F_{(24)} = F_{(34)} = F_{(234)} = F_{(243)}$
(25) $F = F_{(13)} = F_{(24)} = F_{(13)\,(24)} = F_{(1234)} = F_{(12)\,(34)} = F_{(1432)} = F_{(14)\,(23)}$
(26) $F = F_{(12)} = F_{(34)} = F_{(12)\,(34)} = F_{(1324)} = F_{(13)\,(24)} = F_{(1423)} = F_{(14)\,(23)}$
(27) $F = F_{(14)} = F_{(23)} = F_{(12)\,(34)} = F_{(1423)} = F_{(14)\,(23)} = F_{(1324)} = F_{(13)\,(24)}$

(28) $F = F_{(12)(34)} = F_{(13)(24)} = F_{(23)(14)} = F_{(123)} = F_{(132)} = F_{(124)} = F_{(142)}$
$= F_{(134)} = F_{(143)} = F_{(234)} = F_{(243)}$

(29) $F =$ all

338. **Theorem.** It is evident that every group defines a limitation type for an operation F of some degree.

339. **Definitions.** Suppose that in a process all the elements but two are fixed, and that these two vary subject to the process. Then the ranges of values of these two are said to form an *involution of order one.* If all but three elements are fixed, the ranges of these three form an *involution of order two.* Similar definitions may be given for involutions of higher order.

An involution of order r is often called an *implicit* function of $r+1$ variables. The symbol consisting of the process symbol and the constant elements is called an *operator.*

If in any involution of order one the two elements become identical so that they have the same range, for any given set of constant elements, then this set of constant elements constitutes a *multiplex modulus* for the process.

For example, in multiplication $F.(ab = b)$ when a is 1. A similar definition holds for higher involutions.

If in any involution of order r, the constant terms determine an involution whose terms may be *any* elements of the set, then the constant terms constitute a zero for the process. For example, if $F.(0a = 0)$, for all a, 0 is a zero for multiplication. An infinity is, under this definition, also a zero.

We have seen that there co-exist with any process F certain other correlative processes on the same elements. These give us a set of co-existences called *fundamental* identities; but we may have co-existent processes which are not correlatives. In the most general case let us suppose that we have

$$F'.\,a_{11}\,a_{12}\cdots a_{1n_1} \qquad F''.\,a_{21}\,a_{22}\cdots a_{2n_2}\,\ldots\ldots\, F^{(r-1)}.\,a_{r+1,1}\,a_{r-1,2}\cdots a_{r-1,\,n_{r-1}}$$

and that when these processes exist, then we have $F^{(r)}.\,a_{r1}\,a_{r2}\cdots a_{rn_r}$.
We say that $F^{(r)}$ is the implication of the $r-1$ processes preceding. We enter here upon the study of logic proper. For example, if the processes are

$$F'.ab \qquad F''.bc \qquad F'''.ac$$

we have the ordinary syllogism.

We may symbolize this definition by the statement

$$\Phi.\,F'_{n_1}\,F''_{n_2}\,\ldots\ldots\,F^{(r-1)}_{n_{r-1}}\,F^{(r)}_{n_r}$$

and we see then that the form is again that of a process Φ.

We can not enter on the discussion of these cases beyond the single type we need, called the *associative law.*

Let F be such that for every a, b, c, we have

$$F.abd \qquad F.dce \qquad F.bcg \qquad \text{then } F.age$$

then F is called *associative.* The law is usually written $ab.c = a.bc$

Processes subject to this law are the basis of *associative* algebras.[1]

[1] Cf. SCHROEDER 1; RUSSELL 1, 2; HATHAWAY 1.

PART II. PARTICULAR ALGEBRAS.

XIV. COMPLEX NUMBERS.

340. Definitions. The algebra of ordinary complex numbers possesses two qualitative units, $e_0 = 1$, and e_1, such that

$$e_0^2 = e_0 \qquad e_1 e_0 = e_1 = e_0 e_1 \qquad e_1^2 = -e_0 = -1$$

The field of coordinates is the field of positive and negative numbers. The field naturally admits of addition of the units or marks.

341. Theorem. The characteristic equation of the algebra, as well as the general equation, is

$$\zeta^2 - 2x\zeta + x^2 + y^2 = 0$$

or

$$\begin{vmatrix} xe_0 - \zeta & y \\ -y & xe_0 - \zeta \end{vmatrix} = 0$$

Hence for any two numbers

$$\zeta\sigma + \sigma\zeta - 2x\sigma - 2x'\zeta + 2xx' + 2yy' = 0$$

or

$$\zeta\sigma - x\sigma - x'\zeta + xx' + yy' = 0$$

The characteristic equation is irreducible in the field of coordinates, but in the algebra may be written

$$(\zeta - xe_0 - ye_1)(\zeta - xe_0 + ye_1) = 0$$

The numbers $\zeta = xe_0 + ye_1$ and $K\zeta = \bar{\zeta} = xe_0 - ye_1$ are called *conjugates*.

Hence $\xi^2 - 2x\xi + x^2 + y^2 = 0$ has the two solutions ζ, $K\zeta$, or $(\xi - \zeta)(\xi - K\zeta)$ are its factors.

342. Theorem. If several algebras of this kind are added (in the sense defined by SCHEFFERS) we arrive at a WEIERSTRASS commutative algebra.

343. Theorem. If the coordinates are arithmetical numbers we must write this algebra as a cyclic algebra of four qualitative units

$$e_0 \quad e_1 \quad e_2 \quad e_3$$

where

$$e_1^2 = e_2 \qquad e_1^3 = e_3 \qquad e_1^4 = e_0$$

In this case the units e_0 and e_2 are not independent in the field, and combine, by addition, to give zero,[1] and the algebra is of two dimensions.

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[1] STUDY 8, and references there given; BEMAN 2; BELLAVITIS 1–16; *Bibliography of Quaternions.*

XV. QUATERNIONS.

344. Definition. Quaternions is an algebra whose coordinate field is the field of positive and negative numbers, and whose multiplication table is ($e_0 = 1$)

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

345. Theorem. The characteristic equation is

$$\zeta^2 - 2x_0\zeta + x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

The characteristic function of ξ is

$$\xi^2 - 2x_0\xi + x_0^2 + x_1^2 + x_2^2 + x_3^2$$

If we define the conjugate of ζ by

$$\bar{\zeta} = K\zeta = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3$$

then the characteristic function factors into $(\xi - \zeta)(\xi - K\zeta)$.

346. Theorem. The first derived characteristic function of ξ and σ is

$$(\xi - \zeta)(\sigma - K\zeta') + (\sigma - \zeta')(\xi - K\zeta)$$

This vanishes for $\quad \xi = \zeta \qquad \sigma = \zeta'$

or $\quad \xi = K\zeta \qquad \sigma = K\zeta'$

347. Definition. The scalar of ζ is $S\zeta = x_0 = \frac{1}{2}(\zeta + \bar{\zeta})$. The tensor of ζ is given by $(T\zeta)^2 = \zeta\bar{\zeta} = \bar{\zeta}\zeta$.

348. Theorem. We have

$$\zeta^2 - 2\zeta S\zeta + (T\zeta)^2 = 0$$

$$\xi\tau + \tau\xi - 2\xi S\zeta' - 2\tau S\zeta + 2VV\xi V\zeta' + 2VV\tau V\zeta + 2S\zeta\bar{\zeta'} = 0$$

if $\xi = \zeta$, and $\tau = \zeta'$; or if $\xi = \bar{\zeta}$, $\tau = \bar{\zeta'}$; also $S\zeta\bar{\zeta'} = S\bar{\zeta}\zeta'$.

349. Definitions. The versor of ζ is $U\zeta = \dfrac{\zeta}{T\zeta}$ The vector of ζ is

$$V.\zeta = \zeta - S\zeta = \tfrac{1}{2}(\zeta - \bar{\zeta})$$

350. Theorems. $\zeta = S\zeta + V\zeta \qquad \bar{\zeta} = S\zeta - V\zeta$

$$(T\zeta)^2 = (T\bar{\zeta})^2 = (S\zeta)^2 - (V\zeta)^2 = (S\zeta)^2 + (TV\zeta)^2$$

$$U\zeta = U\bar{\zeta} \qquad U\zeta \,.\, U\bar{\zeta} = 1 \qquad U\zeta = (U\zeta)^{-1}$$

$$S.\zeta\sigma = S\sigma\zeta \qquad S.K\zeta = KS.\zeta = S\zeta$$

If $\quad S\zeta = 0 = S\sigma \qquad \sigma = V\sigma \quad \zeta = V\zeta$

and $\quad V\zeta \,.\, V\sigma = -V\sigma V\zeta + 2S.V\sigma V\zeta$

Also $$KV\zeta = -V\zeta = VK\zeta$$

$$\zeta \,.\, V\sigma\zeta^{-1} = -V\sigma + 2\, S\zeta \,.\, V\sigma \,.\, \zeta^{-1} + 2\,\zeta^{-1}\, S\zeta V\sigma$$
$$= -V\sigma + 2\, S\zeta \,.\, V\frac{V\sigma}{\zeta} + 2V \,.\, \zeta^{-1}\, S \,.\, \zeta V\sigma$$

If α, β, γ are vectors,

$$V \,.\, \alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma \qquad V \,.\, \alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta$$

If δ is a vector,

$$\delta S\alpha\beta\gamma = \alpha S\beta\gamma\delta + \beta S\gamma\alpha\delta + \gamma S\alpha\beta\delta = V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta$$

Also

$$V\, V\alpha\beta\, V\gamma\delta = \delta S\alpha\beta\gamma - \gamma S\alpha\beta\delta = \alpha S\beta\gamma\delta - \beta S\alpha\gamma\delta$$

If a, b, c, d, e are quaternions, let us use the notation

$$B \,.\, ab = \tfrac{1}{2}(ab - ba) \qquad B'.\, ab = bSa - aSb$$
$$B \,.\, abc = S \,.\, aBbc - V(aBbc + bBca + cBab)$$

Then

$$B \,.\, ab = B \,.\, \bar{a}b = V \,.\, Va\,Vb = -B \,.\, ba \qquad B \,.\, bb = 0$$
$$S \,.\, aBbc = S \,.\, Va\,Vb\,Vc = -SbBac = \text{etc.}$$
$$B \,.\, abc = -B \,.\, bac = \text{etc.}$$
$$S \,.\, aBbcd = -S \,.\, bBacd = \text{etc.}$$
$$eS \,.\, aBbcd = aS \,.\, eBbcd - bS \,.\, eBcda + cS \,.\, eBdab - dS \,.\, eBabc$$
$$= -Sde \,.\, Babc + Sae \,.\, Bbcd - Sbe \,.\, Bcda + Sce \,.\, Bdab$$

$$B \,.\, abBcde = \begin{vmatrix} c & d & e \\ Sac & Sad & Sae \\ Sbc & Sbd & Sbe \end{vmatrix}$$

$$B'(BabcBdef) = BefSaBbcd + BfdSaBbce + Bde\, SaBbcf$$
$$B\,(BabcBdef) = -B'efSaBbcd - B'fdSaBbce - B'de\, SaBbcf$$

$$B\,(Babc\; Bdef\; Bghi) = \begin{vmatrix} a & b & c \\ SaBdef & SbBdef & ScBdef \\ SaBghi & SbBghi & ScBghi \end{vmatrix}$$

$$|Saa',\, Sbb',\, Scc'| = -SBabc\; Ba'b'c'$$
$$|Saa',\, Sbb'| = SB'ab\; B'a'b' - SBab\; Ba'b'$$
$$|Saa',\, Sbb',\, Scc',\, Sdd'| = -SaBbcd\; Sa'\, Bb'c'd'$$

The solution of the equation $a_1 p + p a_2 = c$ is

$$p = (a_1^2 + 2a_1\, Sa_2 + a_2\,\bar{a}_1)^{-1}\,(a_1 c + c\bar{a}_2)$$

351. Definition. If the coordinate field contain the imaginary $\sqrt{-1}$, we may have for certain quaternions the equation $q^2 = 0$, whence $q = y\theta$, $\theta^2 = 0$, and y is any scalar. In this case there is an infinity of solutions of the equation in the algebra.

Also if $q^2 - 2x_0 q + x_0^2 = 0$, then $q = x_0 + y\theta$.

The nilpotent θ is always of the form $\alpha + \sqrt{-1}\,\beta$, where

$$\alpha^2 = \beta^2 \qquad S.\alpha\beta = 0$$

Since $\sqrt{-1}$ will not combine with positive or negative numbers by addition, we may say that this algebra is in reality the product[1] of real quaternions and the algebra of complex numbers, giving the multiplication table

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$+e_6$	$-e_5$	$-e_4$
e_4	e_4	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_5	e_5	$-e_4$	e_7	$-e_6$	$-e_1$	$+e_0$	$-e_3$	$+e_2$
e_6	e_6	$-e_7$	$-e_4$	e_5	$-e_2$	$+e_3$	$+e_0$	$-e_1$
e_7	e_7	$+e_6$	$-e_5$	$-e_4$	$-e_3$	$-e_2$	$+e_1$	$+e_0$

with equations of condition

$$e_0 + e_4 = 0 \qquad e_1 + e_5 = 0 \qquad e_2 + e_6 = 0 \qquad e_3 + e_7 = 0$$

This algebra HAMILTON called *Biquaternions.*

352. **Definition.** The algebra λ_{110}, λ_{120}, λ_{210}, λ_{220} also is a form to which real quaternions may be reduced by an imaginary transformation.[2] By a rational transformation this becomes

$$\lambda_{110} + \lambda_{220} \qquad \lambda_{110} - \lambda_{220} \qquad \lambda_{210} + \lambda_{120} \qquad \lambda_{210} - \lambda_{120}$$

References to the literature of quaternions would be too numerous to give in full. They may be found in the *Bibliography of Quaternions.* In particular may be mentioned the works of HAMILTON, TAIT, and JOLY.

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[1] That is, any unit may be represented by a double symbol of two independent entities, the two sets of symbols combining independently.

[2] B. PEIRCE 3.

XVI. ALTERNATE ALGEBRAS.

1. ALGEBRAS OF DEGREE TWO, WITH NO MODULUS.

353. Definition. An alternate algebra is one in which the defining units are subject to the law

$$e_i e_j = - e_j e_i \qquad i \neq j$$

The product $e_i e_i = e_i^2$ is variously defined. In the simplest cases e_i^2 is taken equal to zero.

354. Theorem. When $e_i e_j + e_j e_i = 0, \quad i, j = 1 \ldots . r$, we have

$$e_i^2 = 0 \qquad i = 1 \ldots . r$$
$$\zeta^2 = 0,\ \zeta\sigma + \sigma\zeta = 0 \qquad \text{all values of } \zeta, \sigma$$
$$\zeta\sigma\tau = 0 \qquad \text{all values of } \zeta, \sigma, \tau$$

For

$$(\Sigma x_i e_i)^2 = 0 \qquad (\Sigma x_i e_i)(\Sigma y_i e_i) = -(\Sigma y_i e_i)(\Sigma x_i e_i)$$

and[1]

$$\zeta\sigma . \tau = -\zeta . \tau\sigma = \tau . \zeta\sigma = -\tau . \zeta\sigma = 0$$

355. Theorem. We may therefore select a certain set of $r - m - h$ units, $e_1 \ldots . e_{r-m-h}$, whose products $e_i e_j (i, j = 1 \ldots . r - m - h)$ are such that at least one for each subscript does not vanish; we may then choose for the next m units the m independent non-vanishing products of the first $r - m - h$ units; finally, the last h units may be any numbers independent of each other and the first $r - h$ units. We must have[2]

$$m \leqq \frac{2(r-h) + 1 - \sqrt{8(r-h)+1}}{2}$$

or

$$(r - m - h)^2 - (r + m - h) \geqq 0$$

2. GRASSMANN'S SYSTEM.

356. Definition. The next type of alternate numbers is that of GRASSMANN'S *Ausdehnungslehre.* In this case there are m units which may be called *fundamental generators* of the algebra, $e_1 \ldots . e_m$. For them, but not necessarily for their products, the law $e_i e_j + e_j e_i = 0$ $(i, j = 1 \ldots . m)$ holds. They are associative, and consequently the product of $m + 1$ numbers vanishes. There are $r = 2^m - 1$ products or units, e_i, $e_i e_j$, $e_i e_j e_k$, etc.

This algebra uses certain bilinear expressions called *products*, which do not follow the associative law, and also certain *regressive* products, which do not follow this law, and which are multilinear expressions in the coordinates of the factors.[3]

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[1] SCHEFFERS 3. Cf. CAUCHY 1, 2, 3; SCOTT 1, 2, 3. [2] SCHEFFERS 3.

[3] References are too numerous to be given here. In particular see *Bibliography of Quaternions;* GRASSMANN'S works; SCHLEGEL'S papers; HYDE 1, 2, 3, 4, 5, 6, 7, 8, 9; BEMAN 1; WHITEHEAD 1. Cf. WILSON-GIBBS 1; JAHNKE 1. Works on *Vector Analysis* are related to this subject and the next.

3. CLIFFORD ALGEBRAS.

357. Definition. A type of alternates of much use, and which enables all the so-called products of the preceding class to be expressed easily, is that which may be called CLIFFORD'S Algebras. Any such algebra is defined by m generators $e_1 \ldots . e_m$ with the defining equations

$$e_i^2 = -1 \qquad e_i e_j + e_j e_i = 0 \qquad i, j = 1 \ldots . m; \; i \neq j$$
$$e_i . e_j e_k = e_i e_j . e_k \qquad i, j, k = 1 \ldots . m$$

The order[1] is $r = 2^m$.

358. Theorem. If $m = 2m'$, m' an integer, the CLIFFORD algebra of order $2^m = 4^{m'}$ is the product of m' quaternion algebras. If $m = 2m' + 1$, m' an integer, the CLIFFORD algebra of order $2^m = 2^{2m'+1}$ is the product of m' quaternion algebras and the algebra[2]

$$e_0^2 = e_0 = e_1^2 \qquad e_0 e_1 = e_1 e_0 = e_1$$

359. Definition. Since any product such as

$$e_{i_1} e_{i_2} \ldots . e_{i_1} \ldots . e_{i_2} \ldots .$$

may be reduced by successive transpositions to a product of order two lower for every such pair as $e_{i_1} \ldots . e_{i_1}$, it follows that in the product of n numbers $\zeta_1 \ldots . \zeta_n$, where $\zeta_i = \sum_{j=1}^{m} x_{ij} e_j$, we may write

$$\zeta_1 \zeta_2 \ldots . \zeta_n = V_n . \zeta_1 \ldots . \zeta_n + V_{n-2} . \zeta_1 \ldots . \zeta_n + \ldots .$$
$$= \sum_{s=0}^{<\frac{1}{2}(n+1)} V_{n-2s} \zeta_1 \ldots . \zeta_n$$

The expression V_{n-2s} is the sum of all expressions in the product $\zeta_1 \ldots . \zeta_n$ which reduce to terms of order $n - 2s$. Evidently when n is even, the lowest sum is a scalar, V_0; when n is odd, the lowest sum[3] is V_1.

360. Theorem. To reduce to a canonical (simplified) form any homogeneous function of N of the m units, consisting of terms which are each the product of a scalar into n of the units (of order n, therefore), we proceed thus:

Let q be the given function. Then by transposing the units, we may reduce q to the form

$$q = -q' i_1 + q''$$

where i_1 is any given unit, and q' (of order $n - 1$) is independent of q'' (of order n) and of i_1. We find easily

$$q' = V_{n-1} q i_1 \qquad q'' i_1 = V_{n+1} q i_1$$

Therefore

$$V_1 . q q' = V_1 q \, V_{m-1} q i_1 = \varpi_1 = \Phi (i_1)$$

The linear vector function Φ is self-transverse, has therefore real, mutually orthogonal axes. These are the units to be employed to reduce to the canon-

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[1] For this class see *Bibliography of Quaternions*; in particular CLIFFORD'S works; BEEZ 1; LIPSCHITZ 1; JOLY 6, 12, 25; CAYLEY 6, 7.

[2] TABER 1. [3] JOLY 6.

ical form. For example, if q is of order 2, and the function is the general quadratic for N units, there are $\frac{1}{2}N(N-1)$ binary products. Then

$$q = \varpi_1 i_1 + q'$$

$$-\varpi_1 = V_1 qi \qquad Vq\varpi_1 = -i_1\varpi_1^2 + V_1 q'\varpi = -V_1 q\, V_1 qi_1 = -\Phi i_1$$

If i_1 is an axis of this equation so is ϖ_1. Hence the quadratic takes the form

$$q = a_{12} i_1 i_2 + a_{34} i_3 i_4 + \ldots + a_{2p-1,\,2p}\, i_{2p-1} i_{2p}$$

where $p = \frac{1}{2}N$ or $\frac{1}{2}(N-1)$ as N is even or odd.[1]

361. Definition. Let K change the sign of every unit and reverse every product. Then if q is homogeneous, of order p,

$$K \,.\, q_p = (-)^{\frac{1}{2}p(p+1)} q_p$$

Hence $K \,.\, q_p = \pm q_p$ as $p \equiv 0$ or 3 (mod 4) or $\equiv 1$ or 2 (mod 4).

Let I reverse the order of products, but not change signs, thus

$$I \,.\, q_p = q_p \text{ if } p \equiv 0, \text{ or } 1 \pmod 4$$

$$I \,.\, q_p = -q_p \text{ if } p \equiv 2, \text{ or } 3 \pmod 4$$

Let J change the signs of units but not reverse terms.

362. Theorem. $K \,.\, pq = Kq \,.\, Kp \qquad I \,.\, pq = Iq \,.\, Ip \qquad I = JK = KJ$

$J = KI = IK \qquad K = IJ = JI \qquad I^2 = J^2 = K^2 = 1$

363. Theorem. Let p be of order 2, q of order 3; then

$$p_2 q_3 = V_1 \,.\, p_2 q_3 + V_3 \,.\, p_2 q_3 + V_5 \,.\, p_2 q_3$$

Hence, taking conjugates,

$$-q_3 p_2 = -V_1 \,.\, p_2 q_3 + V_3 \,.\, p_2 q_3 - V_5 \,.\, p_2 q_3$$

and

$$V_1 \,.\, p_2 q_3 + V_5 \,.\, p_2 q_3 = \tfrac{1}{2}(p_2 q_3 + q_3 p_2) \qquad V_3 \,.\, p_2 q_3 = \tfrac{1}{2}(p_2 q_3 - q_3 p_2)$$

This process may be applied to any case.

364. Theorem. Let

$$q = q' + q'' \qquad K \,.\, q = q' - q''$$

$$q \,.\, Kq = q'^2 - q''^2 - (q'q'' - q''q')$$

$$Kq \,.\, q = q'^2 - q''^2 + (q'q'' - q''q')$$

Hence

$$q \,.\, Kq = Kq \,.\, q \text{ if } q'q'' - q''q' = 0$$

Let the parts of q be (according as their order $\equiv 0, 1, 2, 3$ mod 4)

$$q = q_{(0)} + q_{(1)} + q_{(2)} + q_{(3)}$$

Then

$$q'q'' = q_{(0)} q_{(1)} + q_{(3)} q_{(2)} + q_{(0)} q_{(2)} + q_{(3)} q_{(1)}$$

and the condition above reduces to

$$q_{(0)} q_{(1)} - q_{(1)} q_{(0)} = q_{(2)} q_{(3)} - q_{(3)} q_{(2)} \qquad q_{(0)} q_{(2)} - q_{(2)} q_{(0)} = q_{(1)} q_{(3)} - q_{(3)} q_{(1)}$$

or

$$V_{(3)}(q_{(0)} q_{(1)} - q_{(2)} q_{(3)}) = 0 \qquad V_{(0)}(q_{(0)} q_{(2)} - q_{(1)} q_{(3)}) = 0$$

[1] JOLY 6. This reference applies to the following sections.

When this is satisfied

$$qKq = V_{(0)}\,(q_{(0)}^2 - q_{(1)}^2 - q_{(2)}^2 + q_{(3)}^2) + 2\,V_{(3)}\,(q_{(0)}\,q_{(3)} - q_{(1)}\,q_{(2)})$$

This is a scalar if $V_{(0)} = V_0$ and $V_{(3)} = 0$.

365. Theorem. $q \,.\, Iq = Iq \,.\, q$ if

$$V_{(0)}\,(q_{(0)}\,q_{(2)} - q_{(3)}\,q_{(1)}) = 0 = V_{(1)}\,(q_{(0)}\,q_{(3)} - q_{(2)}\,q_{(1)})$$

366. Theorem. Let $P = q\rho q^{-1}$, where q is any number, possibly non-homogeneous. Then $P = V_{(1)} \,.\, P$ if qKq and qIq are scalars. But $V_{(1)}$ may not $= V_1$. For example, let

$$\begin{aligned} q &= \cos u \,.\, i_1 i_2 + \sin u \,.\, i_3 i_4 i_5 i_6 \\ q^{-1} &= -\cos u \,.\, i_1 i_2 + \sin u \,.\, i_3 i_4 i_5 i_6 \\ q^2 &= -\cos^2 u + \sin^2 u + 2 \sin u \cos u\; i_1 i_2 i_3 i_4 i_5 i_6 \end{aligned}$$

$\rho = i_1,\ i_2,\ i_3,\ i_4,\ i_5,\ i_6$, then

$$P = q^2 i_1 \qquad q^2 i_2 \qquad -q^2 i_3 \qquad -q^2 i_4 \qquad -q^2 i_5 \qquad -q^2 i_6$$

which are of the form $V_{(1)} = V_1 + V_5$.

367. Theorem. An operator $q(\,)q^{-1}$ can be found which will convert the orthogonal set $i_1, i_2 \ldots . i_n$ into any other orthogonal set $j_1, j_2 \ldots . j_n$; namely,

$$q = 1 - \Sigma j_s\, i_s + \Sigma\Sigma j_s j_t\, i_t\, i_s - \Sigma\Sigma\Sigma j_s j_t j_u\, i_u\, i_t\, i_s + \ldots .$$

where

$$s, t, u \ldots . = 1, 2 \ldots . n \qquad s \neq t \neq u \neq \ldots .$$

If $q\rho q^{-1} = p\rho p^{-1}$ for all values of ρ, a vector, then q is a scalar multiple of p. q may be written

$$q_{12}\, q_{34} \cdots q_{2l-1,\,2l}$$

where $2l = n$ or $n - 1$ as n is even or odd, and

$$q_{12} = \cos \tfrac{1}{2} u_{12} + i_1 i_2 \sin \tfrac{1}{2} u_{12}, \text{ etc.}$$

XVII. BIQUATERNIONS OR OCTONIONS.

368. Definition. Besides HAMILTON's biquaternions, two algebras have received this name. One is the product of real quaternions and the algebra C_1: $e_0^2 = e_0 = e_1^2$, $e_0 e_1 = e_1 e_0 = e_1$; the other is the product of real quaternions and the algebra[1] C_2: $e_0^2 = e_0$, $e_0 e_1 = e_1 e_0 = e_1$, $e_1^2 = 0$.

369. Definitions. Let $\Omega^2 = 0$; let q, r be real quaternions; Ω is commutative with all numbers; $q = \omega + x$; $r = \sigma + y$. Then the octonion Q is given by

$$Q = q + \Omega r$$

We call q the *axial* of Q, Ωr the *converter* of Q. The *axis-direction* of Q is UVq. The *perpendicular* of Q is $\varpi = V \,.\, \sigma\omega^{-1}$. The *rotor* of Q is Vq; the *lator* is Vr; the *motor*, $Vq + \Omega Vr$. The *ordinary scalar* is Sq; the *scalar-converter* is ΩSr; the *convert* is Sr.

We write

$$\begin{array}{llll} & M_1 Q = Vq & M_2 Q = \Omega Vr & MQ = M_1 Q + M_2 Q \\ m \,.\, Q = Vr & S_1 Q = Sq & S_2 Q = \Omega Sr & SQ = S_1 Q + S_2 Q \\ s \,.\, Q = Sr & M \,.\, Q = M_1 Q + \Omega m Q & & S \,.\, Q = S_1 Q + \Omega s Q \end{array}$$

Let $\bar{q}$, $\bar{r}$, $\bar{Q}$ be the conjugates of q, r, Q, also designated by Kq, Kr, KQ. We define

$$KQ = Kq + \Omega Kr, \text{ or } \bar{Q} = \bar{q} + \Omega \bar{r}$$

The *tensors* of q and r are Tq, Tr; the *versors*, Uq, Ur:
The *augmenter* of Q is $TQ = Tq\,(1 + \Omega Srq^{-1}) = T_1 Q \,.\, T_2 Q = T_1 Q\,(1 + \Omega t Q)$.
The *tensor* of Q is $T_1 Q$.
The *additor* of Q is $T_2 Q = 1 + \Omega Srq^{-1}$.
The *pitch* of Q is $tQ = S \,.\, rq^{-1}$. $\quad T_2 Q = 1 + \Omega t Q$.
The *twister* of Q is $UQ = Uq\,(1 + \Omega Vrq^{-1}) = U_1 Q \,.\, U_2 Q$.
The *versor* of Q is $U_1 Q = Uq$.
The *translator* of Q is $U_2 Q = 1 + \Omega Vrq^{-1}$.

Hence

$$Q = T_1 Q \,.\, T_2 Q \,.\, U_1 Q \,.\, U_2 Q$$

370. Theorem. Octonions may be combined under all the laws of quaternions, regard being given to the character of Ω.

371. Theorem. If Q, R be given octonions

$$Q + R = X \qquad QR = Y$$

and if ε is any lator; then if

$$\begin{array}{ll} Q' = Q + \Omega M\varepsilon MQ & R' = R + \Omega M\varepsilon MR \\ X' = X + \Omega M\varepsilon MX & Y' = Y + \Omega M\varepsilon MY \end{array}$$

then

$$Q' + R' = X' \qquad Q'R' = Y'$$

[1] CLIFFORD 1, 4; M'AULAY 2, which applies to sections following; COMBEBIAC 1, 2; STUDY 4, 5

Hence if

$$\phi_\epsilon () = 1 () + \Omega M \varepsilon M ()$$

the application of ϕ_ϵ to all octonions gives an isomorphism of the group of all octonions with itself.

372. Theorem. If $Q = \phi_\rho \,.\, Q'$, or $q + \Omega r = \phi_\rho (q' + \Omega r')$, then

$$q' = q \qquad\qquad r' = r - M(\rho\, Mq)$$

373. Definition. The axial

$$q_Q = q + \Omega\, (Mq)^{-1}\, MMq\, Mr = x + \omega + \Omega\omega^{-1} M\omega\sigma$$

is called the *special axial* of Q, and

$$r_Q = Sr + (1 + \Omega MMr\, (Mq)^{-1})\, Mq\, S\, Mr\, (Mq)^{-1}$$
$$= y + \left(1 + \Omega MM \frac{\sigma}{\omega}\right)\, \omega S\, \frac{\sigma}{\omega}$$

is called the *special convertor-axial* of Q.

374. Theorem. We have

$$Q = q + \Omega r = q_Q + \Omega r_Q$$

where

$$q_Q = \{1 + M\,.\,(Mr\,[Mq]^{-1})\,()\}\, q \;= \phi_{M.\sigma\omega^{-1}} q$$
$$r_Q = \{1 + M\,.\,(Mr\,[Mq]^{-1})\,()\}\, (r - M\,.\,(Mr\,[Mq]^{-1})\, Mq)$$
$$= \phi_{M.\sigma\omega^{-1}}\, (r + M\,.\,\omega M\sigma\omega^{-1}) = \phi_{M.\sigma\omega^{-1}}\, (Sr - \omega^{-1}\, S\omega\sigma)$$

That is

$$Q = \phi_{M.\sigma\omega^{-1}}\, Q', \text{ where } Q' = q + \Omega\, (Sr - \omega^{-1}\, S\omega\sigma)$$

or

$$Q' = q + \Omega\, [Sr - (Mq)^{-1}\, SMq\, Mr]$$

375. Theorem. Any octonion may be considered to be the quotient of two motors. That is, if Q be an octonion it may be written $Q = BA^{-1}$ or $QA = B$, where A and B are a pair of motors.

376. Theorem. $Q^{-1} = q^{-1} - \Omega\, q^{-1}\, r\, q^{-1}$, when $q \neq 0$.

377. Definition. The angle of q is *the angle* of Q.

378. Theorem. $Q\; ()\; Q^{-1}$ produces from the operand a new operand which has been produced from the first by rotating it as a rigid body about the axis of Q through twice the angle of Q, and translated through twice the translation of Q.

379. Theorem. If A and B are motors

$$A = \alpha_1 + \Omega\, \alpha_2 = (1 + \Omega p)\, \alpha_1$$
$$B = \beta_1 + \Omega\, \beta_2 = (1 + \Omega p')\, \beta_1 = \{1 + \Omega\, (\varpi + p')\}\, \beta$$

and

$$d = \pm T\varpi, \quad \theta = \angle \frac{\beta}{\alpha_1}$$

then

$$AB = \alpha_1 \beta + \Omega\, (p + p' - \varpi)\, \alpha_1 \beta$$
$$M\,.\,AB = M\alpha_1\beta + \Omega\, \{(p + p')\, M\alpha_1\beta - \varpi\, S\alpha_1\beta\}$$
$$M_1\, AB = M\alpha_1\beta \qquad m\,.\,AB = (p + p')\, M\alpha_1\beta - \varpi\, S\alpha_1\beta$$
$$t M\,.\,AB = p + p' - \varpi\, M^{-1}\alpha_1\beta\, S\alpha_1\beta = p + p' + d\cot\theta$$

Hence axis $M\,.\,AB$ is ϖ, pitch $= p + p' + d\cot\theta$

If A and B are parallel, we determine $M.AB$ by

$$M.AB = -\Omega\varpi\alpha_1\beta$$

Again $$S.AB = S.\alpha_1\beta + \Omega\{(p+p')S\alpha_1\beta - \varpi M\alpha_1\beta\}$$

$$S_1 AB = S\alpha_1\beta$$

$$s.AB = (p+p')S\alpha_1\beta - \varpi M\alpha_1\beta$$

$$tS.AB = p + p' - \varpi M\alpha_1\beta S^{-1}\alpha_1\beta = p + p' - d\tan\theta$$

$$M_1.AB + S_1.AB = \alpha_1\beta$$

$$m\,AB + s\,AB = (p + p' - \varpi)\alpha_1\beta$$

$$t\,AB = p + p' \qquad u.AB = -\varpi$$

$$T_1.AB = T(\alpha_1\beta) \quad U_1 AB = U(\alpha_1\beta)$$

For the sum we have

$$A + B = \{1 + \Omega(p'' + \varpi')\}(\alpha_1 + \beta)$$

where $$p'' + \varpi' = (p\alpha_1 + p'\beta + \varpi\beta)(\alpha_1 + \beta)^{-1}$$

or

$$p'' = S(p\alpha_1 + p'\beta)(\alpha_1 + \beta)^{-1} - \varpi M\alpha_1\beta.(\alpha_1 + \beta)^{-2}$$

$$\varpi' = \varpi S\beta(\alpha_1 + \beta)^{-1} + (p - p')M\alpha_1\beta.(\alpha_1 + \beta)^{-2}$$

380. Theorem. If A, B, C be three motors, and if d and θ are defined as in §379 for A, B, and likewise e, ϕ are corresponding quantities for $M.AB$ and C, then

$$tSABC = tA + tB + tC + d\cot\theta - e\tan\phi$$

Hence if we have three motors 1, 2, 3, and if the distances and angles are: for $23: d_1, \theta_1$; for $31: d_2, \theta_2$; for $12: d_3, \theta_3$, and for 1 and $d_1: e_1, \phi_1$; 2 and $d_2: e_2, \phi_2$; 3 and $d_3: e_3, \phi_3$, then

$$d_1\cot\theta_1 - e_1\tan\phi_1 = d_2\cot\theta_2 - e_2\tan\phi_2 = d_3\cot\theta_3 - e_3\tan\phi_3$$

381. Theorem.

$$T_1(QR\ldots) = T_1Q.T_1R\ldots \qquad t(QR\ldots) = tQ + tR + \ldots$$

$$S_1^2Q - M_1^2Q = T_1^2Q \qquad tSQ.S_1^2Q - tMQ.M_1^2Q = tQ.T_1^2Q$$

and $$tMQ = \frac{tQ\,T_1^2Q - tSQ.S_1^2Q}{T_1^2Q - S_1^2Q}$$

382. Theorem.

$$tM.ABC = tA + tB + tC - \frac{d\cot\theta - e\tan\phi}{\cot^2\theta\tan^2\phi + \cot^2\theta + \tan^2\phi}$$

$$tM.(MAB)C = tA + tB + tC + d\cot\theta + e\cot\phi$$

383. Theorem. If E is coaxial with A, B, C, then

$$ES.ABC = AS.BCE + BS.CAE + CS.ABE$$
$$= MBC.SAE + MCA.SBE + MAB.SCE$$

384. Theorems.

$$S\,.\,(Q+R)=SQ+SR \qquad S_1(Q+R)=S_1Q+S_1R$$

$$S_2(Q+R)=S_2Q+S_2R \qquad s(Q+R)=sQ+sR$$

$$s\Omega Q=S_1Q \qquad S_2\Omega Q=\Omega S_1 Q \qquad S_1\Omega Q=0 \qquad \Omega S_2 Q=0$$

$$SQ=s\Omega Q+\Omega s Q$$

If $i\perp j\perp k$, then

$$A=-iSiA-jSjA-kSkA$$

or

$$A=-is\Omega iA-js\Omega jA-ks\Omega kA-\Omega isiA-\Omega jsjA-\Omega kskA$$

If

$$A=xi+yj+zk+l\Omega i+m\Omega j+n\Omega k$$

and

$$\vartheta=i\frac{\partial}{\partial l}+j\frac{\partial}{\partial m}+k\frac{\partial}{\partial n}+\Omega i\frac{\partial}{\partial x}+\Omega j\frac{\partial}{\partial y}+\Omega k\frac{\partial}{\partial z}$$

then

$$(s\,.\,dA\vartheta)=-d()$$

ϑ is independent of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

If A is a lator, $s\,.\,A^2=0$.

If A is not a lator

$$s\,.\,A^2=2tAM_1^2A=-2tAT_1^2A \qquad s(TQ)^2=2tQ\,.\,T_1^2Q$$

385. Definitions. The motors A_1, $A_2\ldots.A_n$ are *independent* when no relation exists of the form

$$x_1A_1+\ldots.+x_nA_n=0, \qquad [x_1\ldots.x_n \text{ scalars}]$$

If independent, the motors $x_1A_1+\ldots.+x_nA_n=\Sigma xA$ form a *complex* of order n, called the complex of $A_1\ldots.A_n$. The complex of highest order is the sixth, to which all motors belong.

Two motors A_1, A_2 are *reciprocal* if $sA_1A_2=0$. The n motors $A_1\ldots.A_n$ are *co-reciprocal* if every pair is a reciprocal pair; in such case A_1 is reciprocal to every motor of the complex $A_2\ldots.A_n$, and every motor of the complex $A_1\ldots.A_r$ to every one of the complex $A_{r+1}\ldots.A_n$. The only self-reciprocal motors are lators and rotors. Of six independent co-reciprocal motors none is a lator or a rotor.

386. Theorem. If A, B, C are motors, $S\,.\,ABC=0$ if and only if

(1) Two independent motors of the complex A, B, C are lators, or

(2) $XA+YB+ZC=0$, where X, Y, Z are scalar octonions whose ordinary scalar parts are not all zero.

387. For linear octonion functions and octonion differentiation reference may be made to M'AULAY's text.[1]

[1] M'AULAY 2.

XVIII. TRIQUATERNIONS AND QUADRIQUATERNIONS.

388. Definition. Triquaternions is an algebra which is the product of quaternions and the algebra[1]

$$\omega^2 = 0 \qquad \mu^2 = 1 \qquad \omega\mu = -\mu\omega = \omega$$

389. Definition. If $r = w + \rho + \omega(w_1 + \rho_1) + \mu(w_2 + \rho_2) = q + \omega q_1 + \mu q_2$, where q, q_1, q_2 are ordinary quaternions, then we write and define

$$r = w + (\omega w_1 + \mu\rho_2) + (\mu w_2 + \rho + \omega\rho_1) = G\,.\,r + L\,.\,r + P\,.\,r = w + l + p$$

where

$G\,.\,r = w = S\,.\,q$

$L\,.\,r = \mu w_2 + \rho + \omega\rho_1 = \mu Sq_2 + Vq + \omega Sq_1$, called a *linear element;*

$P\,.\,r = \omega w_1 + \mu\rho_2 = \omega Sq_1 + \mu Vq_2$, called a *plane.*

Further, we write

$$L\,.\,r = (\mu w_2 + \omega\beta) + (\rho + \omega\rho_1 - \omega\beta)$$

where we determine β by the equation

$$(w_2^2 - \rho^2)\,\beta = w_2^2\rho + w_2 V\rho\rho_1 - \rho S\rho\rho_1$$

then we define

$$m = (\mu w_2 + \omega\beta), \text{ called a } \textit{point}$$

$$d = (\rho + \omega\rho_1 - \omega\beta), \text{ called a } \textit{line}$$

$$L\,.\,r = m + d$$

We define further

$$\overline{L}\,.\,r = m - d, \text{ the conjugate of } L\,.\,r$$

390. Theorem.

$$\begin{array}{lll} G\,.\,ll' = G\,.\,l'l & L\,.\,ll' = -L\,.\,l'l & P\,.\,ll' = P\,.\,l'l \\ G\,.\,lp = 0 & L\,.\,lp = L\,.\,pl & P\,.\,lp = -P\,.\,pl \\ G\,.\,pp' = G\,.\,p'p & L\,.\,pp' = -L\,.\,p'p & P\,.\,pp' = 0 \end{array}$$

391. Theorem.

$$G\,.\,md = G\,.\,dm = 0 \qquad P\,.\,md = P\,.\,dm \qquad L\,.\,md = -L\,.\,dm = 0$$

392. Theorem. $Lr\,.\,\overline{L}r = m^2 - d^2 \qquad l^{-1} = \dfrac{1}{m^2 - d^2}\,\overline{l}$

393. Definitions. $T\,.\,r = \sqrt{w^2 + l\overline{l} - p^2}$

If

$$Vq_2 = 0 \qquad P\,.\,r = \omega Sq_1 = \omega \mathrm{P}\,.\,r$$

or

$$\mathrm{P}\,.\,r = Sq_1, \text{ if } Vq_2 = 0$$

394. Theorem. Let

$$A = w^2 + l\overline{l} - p^2 = q\overline{q} + q_2\overline{q}_2$$

$$B = 2\,(wTm - TLpd) = q\overline{q}_2 + q_2\overline{q}$$

then

$$r^{-1} = (A^2 - B^2)\,[(A - \mu B)\,(\overline{q} + \mu\overline{q}_2) - \omega\,(\overline{q} - \overline{q}_2)\,q_1\,(\overline{q} + \overline{q}_2)]$$

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[1] COMBEBIAC 2. This reference applies to the following sections.

395. Definition. Let

$$m = \mu x_0 + \omega\rho \qquad m' = \mu x_0' + \omega\rho' \qquad c = \alpha + \omega\beta$$

Then we define

$$V \,.\, m, m' = x_0\rho' - x_0'\rho + \omega L\rho\rho'$$
$$S \,.\, c, m = \mu(x_0\beta + G\alpha\rho) + \omega G\beta\rho$$
$$S \,.\, m, m', m'' = S(V \,.\, m, m')m'' = S \,.\, mV \,.\, m', m''$$

396. Theorem. $V \,.\, \omega\rho, \omega\rho' = \omega L\rho\rho' \qquad S \,.\, \omega\rho, \omega\rho' = \omega G\rho\rho'$

$$S \,.\, c, m = S \,.\, m, c \qquad V \,.\, m, m' = - Vm', m$$

$$G \,.\, cS \,.\, c, m = 0 \qquad L \,.\, cS \,.\, c, m = \tfrac{1}{2}Pc^2 \,.\, m \qquad P \,.\, cS \,.\, c, m = S \,.\, c, Lcm$$
$$G \,.\, S \,.\, c, m \,.\, m = 0 \qquad L \,.\, S \,.\, c, m \,.\, m = V \,.\, m, Lcm \qquad P \,.\, S \,.\, c, m \,.\, m = 0$$
$$G \,.\, mV \,.\, m, m' = 0 \qquad L \,.\, mVm, m' = 0 \qquad P \,.\, mV \,.\, m, m' = -S \,.\, m, Lmm'$$

397. Theorem.

$$Gll' = x_0x_0' + S\rho\rho'$$
$$Lll' = V\rho\rho' + \omega\,[V(\rho\rho_1' + \rho_1\rho') + x_0'\rho_1 - x_0\rho_1']$$
$$Pll' = \mu(x_0\rho' + x_0'\rho) + \omega S(\rho\rho_1' + \rho_1\rho')$$
$$Lmm' = x_0'\rho_1 - x_0\rho_1'$$

398. Theorem. $Gpp' = -Tp\,Tp'\cos(p, p') \qquad Lpp' = Tp\,Tp'\,\delta\sin(p, p')$

399. Theorem. $l = \mu x_0 + \rho + \omega\rho_1 \qquad p = \mu\alpha + \omega w \qquad Glp = 0$

$$Llp = \mu S\rho\alpha + x_0\alpha + \omega(w\rho + V\rho_1\alpha)$$

Let μ, ω, ω' be units satisfying the multiplication table

	μ	ω	ω'
μ	1	$-\omega$	ω'
ω	ω	0	$2(\mu - 1)$
ω'	$-\omega'$	$-2(\mu + 1)$	0

and let the *quadriquaternion*[1] Λ be defined by the equation

$$\Lambda = q + \mu q_1 + \omega q_2 + \omega' q_3$$

where q, q_1, q_2, q_3 are real quaternions. The units μ, ω, ω' are commutative with q, q_1, q_2, q_3. If $q_3 = 0$, Λ becomes a triquaternion.

We may write Λ as the sum of three parts each of which may be found uniquely:

$$\Lambda = G \,.\, \Lambda + L \,.\, \Lambda + P \,.\, \Lambda$$

where

$$G \,.\, \Lambda = S \,.\, q$$
$$L \,.\, \Lambda = V \,.\, q + \mu S \,.\, q_1 + \omega V \,.\, q_2 + \omega' V \,.\, q_3$$
$$P \,.\, \Lambda = \mu V \,.\, q_1 + \omega S \,.\, q_2 + \omega' S \,.\, q_3$$

Then the formulæ of §390 above hold for quadriquaternions as well as for triquaternions, if $l = L \,.\, \Lambda$, $p = P \,.\, \Lambda$, etc.

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[1] Combebiac 3.

XIX. SYLVESTER ALGEBRAS.

1. NONIONS.

400. Definition. Nonions is the quadrate algebra of order 9, corresponding to quaternions, which is of order 4. In one form its units are[1]

$$\lambda_{110} \quad \lambda_{120} \quad \lambda_{130} \quad \lambda_{210} \quad \lambda_{220} \quad \lambda_{230} \quad \lambda_{310} \quad \lambda_{320} \quad \lambda_{330}$$

401. Theorem. The nonion units may be taken in the forms (irrational transformation in terms of ω, a primitive cube root of unity)

$$\begin{array}{lll} e_0 = 1 = \lambda_{110} + \lambda_{220} + \lambda_{330} & i = \lambda_{110} + \omega\lambda_{220} + \omega^2\lambda_{330} & i^2 = \lambda_{110} + \omega^2\lambda_{220} + \omega\lambda_{330} \\ j = \lambda_{120} + \lambda_{230} + \lambda_{310} & j^2 = \lambda_{130} + \lambda_{210} + \lambda_{320} & ij = \lambda_{120} + \omega\lambda_{230} + \omega^2\lambda_{310} \\ ij^2 = \lambda_{130} + \omega\lambda_{210} + \omega^2\lambda_{320} & i^2j = \lambda_{120} + \omega^2\lambda_{230} + \omega\lambda_{310} & i^2j^2 = \lambda_{130} + \omega^2\lambda_{210} + \omega\lambda_{320} \end{array}$$

whence[2]

$$\begin{array}{llllll} i^3 = 1 & j^3 = 1 & (ij)^3 = 1 & (ij^2)^3 = 1 & (i^2j)^3 = 1 & (i^2j^2)^3 = 1 \\ ji = \omega ij & ji^2 = \omega^2 i^2 j & j^2 i = \omega^2 ij^2 & j^2 i^2 = \omega i^2 j^2 & & \end{array}$$

402. Theorem. If

$$\phi = \Sigma x_{ab}\, i^a j^b \qquad a, b = 0, 1, 2$$

Then

$$\begin{array}{lll} S.\ \phi = x_{00} & S.i\ \phi = x_{20} & S.i^2\ \phi = x_{10} \\ S.j\ \phi = x_{02} & S.ij\ \phi = \omega^2 x_{22} & S.i^2j\ \phi = \omega\, x_{12} \\ S.j^2\phi = x_{01} & S.ij^2\phi = \omega x_{21} & S.i^2j^2\phi = \omega^2 x_{11} \end{array}$$

$$\begin{aligned} S.\phi\psi &= S(\Sigma x_{ab}\, i^a j^b)(\Sigma y_{cd}\, i^c j^d) = S\Sigma\, x_{ab}\, y_{cd}\, \omega^{bc}\, i^{a+c} j^{b+d} \left(\begin{smallmatrix} a+c\equiv 0 \\ b+d\equiv 0\end{smallmatrix}\right) \pmod 3 \\ &= (x_{00}\,y_{00} + x_{10}\,y_{20} + x_{20}\,y_{10} + x_{01}\,y_{02} + \omega^2 x_{11}\,y_{22} + \omega x_{21}\,y_{12} \\ &\qquad + x_{02}\,y_{01} + \omega x_{12}\,y_{21} + \omega^2 x_{22}\,y_{11}) \end{aligned}$$

Hence

$$S.\phi\psi = S.\psi\phi \qquad S.\phi\psi\chi = S.\chi\phi\psi = S.\psi\chi\phi$$

and if

$$\phi = \psi \qquad S.i^a j^b \phi = S.i^a j^b \psi$$

therefore

$$x_{ab} = y_{ab}$$

$$\phi\psi = \Sigma\, x_{ab}\, y_{cd}\, \omega^{bc}\, i^{a+c}\, j^{b+d} \qquad a, b, c, d = 0, 1, 2$$

$$\phi\psi\chi = \Sigma\, x_{ab}\, y_{cd}\, z_{ef}\, \omega^{bc+be+de}\, i^{a+c+e}\, j^{b+d+f} \qquad a, b, c, d, e, f = 0, 1, 2$$

403. Definition. If $S.j\phi = 0 \qquad S.j^2\phi = 0 \qquad S.j = 0$ then we define

$$K_j.\phi = j\phi j^{-1} \qquad K_j^2.\phi = j^2\phi j^{-2}$$

404. Theorem. We may write ϕ in the form $\phi = a + bi + ci^2$ (at least if ϕ has not equal roots); whence, if j is chosen,[3] so that $Sj = 0$, $S.ji = 0$, $Sj^2i = 0$, we have

$$K_j\phi = j\phi j^{-1} = a + \omega bi + \omega^2 ci^2 \qquad K_j^2\phi = j^2\phi j^{-2} = a + \omega^2 bi + \omega ci^2$$

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[1] SYLVESTER 3, 4; TABER 2; C. S. PEIRCE 6; also the linear vector operator in space of three dimensions, *Bibliography of Quaternions*, in particular HAMILTON, TAIT, JOLY, SHAW 2; also articles on matrices.

[2] SHAW 7. This applies to §§ 402–403.

[3] Cf. TABER 2.

405. Theorem. If j' is to be such that $S.j' = 0$, $Sij' = 0$, $Si^2j' = 0$, and if j is such that $S.j = 0$, $Sij = 0$, $Sij^2 = 0$, we may take

$$j' = \alpha_1 j + \alpha_2 j^2 + \beta_1 ij + \beta_2 ij^2 + \gamma_1 i^2 j + \gamma_2 i^2 j^2$$

whence

$$j'^2 = 2\alpha_1\alpha_2 + 2\beta_1\gamma_2\omega^2 + 2\beta_2\gamma_1\omega + i(\beta_1\alpha_2 + \omega^2\beta_1\alpha_2 + \alpha_1\beta_2\omega + \alpha_1\beta_2)$$
$$+ i^2(\alpha_1\gamma_2 + \alpha_1\gamma_2\omega^2 + \alpha_2\gamma_1\omega + \alpha_2\gamma_1) + \ldots.$$

and if $S.j'^2 = 0$, $Sij'^2 = 0$, $Si^2j' = 0$ then

$$\alpha_1\alpha_2 + \omega\beta_2\gamma_1 + \omega^2\beta_1\gamma_2 = 0 \qquad \beta_1\alpha_2 + \omega\alpha_1\beta_2 = 0 \qquad \alpha_1\gamma_2 + \omega\alpha_2\gamma_1 = 0$$

whence

$$\alpha_1^2 = 2\beta_1\gamma_1 \qquad \alpha_2^2 = 2\beta_2\gamma_2$$

and

$$\alpha_2 : \beta_2 : \gamma_2 = \alpha_1 : -\omega^2\beta_1 : -\omega\gamma_1 \text{ or } \alpha_2 = \beta_2 = \gamma_2 = 0$$

That is

$$j' = (\alpha_1 + \beta_1 i + \gamma_1 i^2)j = j(\alpha_1 + \omega^2\beta_1 i + \omega\gamma_1 i^2)$$

Hence

$$j'^{-1} = j^2(\alpha_1 + \beta_1 i + \gamma_1 i^2)^{-1} = (\alpha_1 + \omega^2\beta_1 i + \omega\gamma_1 i^2)j^{-1}$$

and

$$j'(a + bi + ci^2)j'^{-1} = j(a + bi + ci^2)j^{-1}$$

It is thus immaterial what vector j we take to produce the conjugate $K_j\phi$, except that we cannot discriminate between $K_j\phi$ for one vector and $K_{j'}^2\phi$ for another, if the second is equivalent to the square of the first. We may therefore omit the subscript j and write simply K, K^2.

406. Theorem. From $\phi = a + bi + ci^2$ we have

$$\phi^3 - 3a\phi^2 + 3(a^2 - bc)\phi - (a^3 + b^3 + c^3 - 3abc) = 0$$

or

$$\begin{vmatrix} S.\phi - \phi & S.i\phi & S.i^2\phi \\ S.i^2\phi & S.\phi - \phi & S.i\phi \\ S.i\phi & S.i^2\phi & S.\phi - \phi \end{vmatrix} = 0$$

407. Theorem.[1]

$$\phi + K\phi + K^2\phi = 3S\phi = T_1\phi$$
$$\phi K\phi + \phi K^2\phi + K\phi K^2\phi = 3(S^2\phi - Si\phi\, Si^2\phi) = T_2\phi$$
$$\phi K\phi K^2\phi = S^3\phi + S^3 i\phi + S^3 i^2\phi - 3\, S\phi\, Si^2\phi\, Si\phi = T_3\phi$$

$$T_1\phi = T_1 K\phi = T_1 K^2\phi \qquad T_2\phi = T_2 K\phi = T_2 K^2\phi \qquad T_3\phi = T_3 K\phi = T_3 K^2\phi$$

408. Theorem. If $\alpha = 1 + i + i^2$, where $\phi = \Sigma x_{ab}\, i^a j^b$,

$$\begin{vmatrix} S.\alpha\phi - \phi & S.j^{-1}\alpha\phi & S.j^{-2}\alpha\phi \\ S.\alpha j\phi & S.j^{-1}\alpha j\phi - \phi & S.j^{-2}\alpha j\phi \\ S.\alpha j^2\phi & S.j^{-1}\alpha j^2\phi - \phi & S.j^{-2}\alpha j^2\phi - \phi \end{vmatrix} = 0$$

[1] Cf. TABER 2.

Hence

$$T_1\phi = S(\alpha + j^{-1}\alpha j + j^{-2}\alpha j^2)\phi = S(\alpha + K\alpha + K^2\alpha)\phi$$
$$T_2\phi = (S\alpha\phi\, SK\alpha\phi + SK\alpha\phi\, SK^2\alpha\phi + S\alpha\phi\, SK^2\alpha\phi$$
$$- S\alpha j\phi\, Sj^{-1}\alpha\phi - S\alpha j^2\phi\, Sj^{-2}\alpha\phi - Sj^{-1}\alpha j^2\phi\, Sj^{-2}\alpha j\phi)$$
$$T_3\phi = \begin{vmatrix} S\alpha\phi & Sj^{-1}\alpha\phi & Sj^{-2}\alpha\phi \\ S\alpha j\phi & Sj^{-1}\alpha j\phi & Sj^{-2}\alpha j\phi \\ S\alpha j^2\phi & Sj^{-1}\alpha j^2\phi & Sj^{-2}\alpha j^2\phi \end{vmatrix}$$

409. Theorem.[1] $\phi^3 - 3S\phi\,.\,\phi^2 + \frac{3}{2}(3S^2\phi - S\phi^2)\,\phi - (\frac{9}{2}S^3\phi - \frac{9}{2}S\phi^2 S\phi + S\phi^3) = 0$

$$\phi_1^2\phi_2 + \phi_1\phi_2\phi_1 + \phi_1\phi_2^2 - 3S\phi_1\,.\,(\phi_1\phi_2 + \phi_2\phi_1) - 3S\phi_2\,.\,\phi_1^2$$
$$+ 3(S^2\phi_1 - \tfrac{1}{2}SV^2\phi_1)\phi_2 + 3(2S\phi_1 S\phi_2 - SV\phi_1 V\phi_2)\,\phi_1$$
$$- (3S^2\phi_1 S\phi_2 + 3SV^2\phi_1 V\phi_2 - 3S\phi_1 SV\phi_1 V\phi_2 - \tfrac{3}{2}S\phi_2 SV^2\phi_1) = 0$$

where
$$V\phi = (\phi - S\phi)$$

Also

$$\phi_1\phi_2\phi_3 + \phi_1\phi_3\phi_2 + \phi_2\phi_1\phi_3 + \phi_2\phi_3\phi_1 + \phi_3\phi_1\phi_2 + \phi_3\phi_2\phi_1$$
$$- 3S\phi_1\,.\,(\phi_2\phi_3 + \phi_3\phi_2) - 3S\phi_2\,.\,(\phi_1\phi_3 + \phi_3\phi_1) - 3S\phi_3\,.\,(\phi_1\phi_2 + \phi_2\phi_1)$$
$$+ 3S\phi_1(3S\phi_2 S\phi_3 - S\phi_2\phi_3) + 3\phi_2(3S\phi_1 S\phi_3 - S\phi_1\phi_3)$$
$$+ 3\phi_3(3S\phi_1 S\phi_2 - S\phi_1\phi_2) - (27S\phi_1 S\phi_2 S\phi_3 - 9S\phi_1 S\phi_2\phi_3$$
$$- 9S\phi_2 S\phi_1\phi_3 - 9S\phi_3 S\phi_1\phi_2 + 3S\phi_1\phi_2\phi_3 + 3S\phi_1\phi_3\phi_2) = 0$$

410. Theorem. If

$$f_k(\theta, \eta) = \tfrac{1}{3}\left[e^{\theta+\eta} + \omega^k e^{\omega\theta+\omega^2\eta} + \omega^{2k} e^{\omega^2\theta+\omega\eta}\right] \qquad k = 0, 1, 2$$

then

$$\phi = (T_3\phi)^{\frac{1}{3}}[f_0(\theta,\eta) + i f_1(\theta,\eta) + i^2 f_2(\theta,\eta)]$$
$$= (T_3\phi)^{\frac{1}{3}}[f_0(\theta,0) + i f_1(\theta,0) + i^2 f_2(\theta,0)]\,[f_0(\eta,0) + i f_1(\eta,0) + i^2 f_2(\eta,0)]$$

If ϕ_1 and ϕ_2 have the same unit i,

$$\phi_1 = a + bi + ci^2 \qquad \phi_2 = a' + b'i + c'i^2$$

$$\phi_1\phi_2 = (T_3\phi_1)^{\frac{1}{3}}(T_3\phi_2)^{\frac{1}{3}}[f_0(\theta+\theta', \eta+\eta') + if_1(\theta+\theta', \eta+\eta') + i^2 f_2(\theta+\theta', \eta+\eta')]$$

The functions f_k satisfy the addition formulae[1]

$$f_k(\theta+\theta', \eta+\eta') = f_0(\theta,\eta)f_k(\theta',\eta') + f_1(\theta,\eta)f_{k+2}(\theta',\eta') + f_2(\theta,\eta)f_{k+1}(\theta',\eta')$$
$$f_k(\omega\theta, 0) = \omega^k f_k(\theta, 0) \qquad f_k(\omega\theta, \omega^2\eta) = \omega^k f_k(\theta,\eta) \qquad f_k(\theta, 0) = f_{2k}(0, \theta)$$
$$K\,.\,\phi = (T_3\phi)^{\frac{1}{3}}[f_0(\omega\theta, \omega^2\eta) + if_1(\omega\theta, \omega^2\eta) + i^2 f_2(\omega\theta, \omega^2\eta)]$$
$$K^2\,.\,\phi = (T_3\phi)^{\frac{1}{3}}[f_0(\omega^2\theta, \omega\eta) + if_1(\omega^2\theta, \omega\eta) + i^2 f_2(\omega^2\theta, \omega\eta)]$$
$$\phi^{-1} = (T_3\phi)^{-\frac{1}{3}}[f_0(-\theta, -\eta) + if_1(-\theta, -\eta) + i^2 f_2(-\theta, -\eta)]$$
$$\phi^p = (T_3\phi)^{p/3}[f_0(p\theta, p\eta) + if_1(p\theta, p\eta) + i^2 f_2(p\theta, p\eta)]$$

411. Theorem. The characteristic equation of $\phi = \Sigma\, x_{ab}\, i^a j^b$ may be written

$$\begin{vmatrix} x_{00} + x_{10} + x_{20} - \phi & x_{01} + x_{11} + x_{21} & x_{02} + x_{12} + x_{22} \\ x_{02} + \omega x_{12} + \omega^2 x_{22} & x_{00} + \omega x_{10} + \omega^2 x_{20} - \phi & x_{01} + \omega x_{11} + \omega^2 x_{21} \\ x_{01} + \omega^2 x_{11} + \omega x_{21} & x_{02} + \omega^2 x_{12} + \omega x_{22} & x_{00} + \omega^2 x_{10} + \omega x_{20} - \phi \end{vmatrix} = 0$$

[1] Taber 2.

3. MATRICES AS QUADRATES.

423. Definition. A matrix (as understood here) is a quadrate of any order; that is, a SYLVESTER algebra, usually of order $> 4^2$. Its units are called vids[1] if they take the form

$$\lambda_{ij0} \qquad i, j = 1 \ldots n$$

424. Theorem. The general quadrate may be defined by the $r = n^2$ units

$$e_{ab} \qquad a, b = 1 \ldots n$$

such that

$$e_{ab} = i^a j^b$$

where

$$i = \sum_{s=1}^{n} \omega^s \lambda_{ss0} \qquad j = \lambda_{120} + \lambda_{230} + \ldots + \lambda_{n-1, n, 0} + \lambda_{n10}$$

$$ji = \omega ij \qquad i^n = j^n = (i^a j^b)^n = 1$$

$$\omega = \cos\frac{2\pi}{n} + \sqrt{-1}\,\sin\frac{2\pi}{n}$$

If[2]

$$\phi = \Sigma\, x_{ab}\, i^a j^b \qquad a,\ b = 1 \ldots n$$

$$\psi = \Sigma\, y_{cd}\, i^c j^d \qquad c,\ d = 1 \ldots n$$

$$\phi\psi = \Sigma\, .\, x_{ab}\, y_{cd}\, \omega^{bc}\, i^{a+c} j^{b+d} \qquad a, b, c, d = 1 \ldots n$$

425. Theorem.[3] $S\,.\,\phi = x_{00} \qquad S\,.\,j^{-b} i^{-a} \phi = x_{ab} \qquad S\,.\,\phi\psi = S\,.\,\psi\phi$

426. Theorem. Since every quadrate in the second form may be reduced to the first form, it is easily seen that ϕ satisfies the identity (characteristic equation)

$$\begin{vmatrix} \Sigma\ x_{s0} - \phi & \Sigma\, x_{s1} & \ldots\ldots & \Sigma\, x_{sn-1} \\ \Sigma\, \omega^s x_{sn-1} & \Sigma\, \omega^s x_{s0} - \phi & \ldots\ldots & \Sigma\, \omega^s x_{sn-2} \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma\, \omega^{s(n-1)} x_{s1} & \Sigma\, \omega^{s(n-1)} x_{s2} & \ldots\ldots & \Sigma\, \omega^{s(n-1)} x_{s0} - \phi \end{vmatrix} = 0$$

in each term[4] Σ represents $\sum_{s=0}^{n-1}$

427. Theorem. We may write

$$\phi = \Sigma\, i^a j^b\, S j^{-b} i^{-a} \phi \qquad a, b = 0 \ldots n-1$$

$$S\,.\,\phi\psi = \Sigma\,.\, S i^a j^b \phi\, S j^{-b} i^{-a} \psi \qquad a, b = 0 \ldots n-1$$

If $\alpha = 1 + i + i^2 + \ldots + i^{n-1}$, then the identical equation is

$$\begin{vmatrix} S\alpha\phi - \phi & S j^{-1} \alpha\, \phi & \ldots\ldots & S j^{-(n-1)} \alpha\, \phi \\ S\alpha j\, \phi & S j^{-1} \alpha j\, \phi - \phi & \ldots\ldots & S j^{-(n-1)} \alpha j\, \phi \\ \ldots & \ldots & \ldots & \ldots \\ S\alpha j^{n-1} \phi & S j^{-1}\, \alpha j^{n-1}\ \phi & \ldots\ldots & S j^{-(n-1)} \alpha j^{(n-1)} \phi - \phi \end{vmatrix} = 0$$

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[1] LAGUERRE 1; CAYLEY 5; B. PEIRCE 3; C. S. PEIRCE, 4, 8; STEPHANOS 1; TABER 1; SHAW 7; LAURENT 1, 2, 3, 4. On the general topic see *Bibliography of Quaternions*.

[2] SHAW 7; LAURENT 1.

[3] TABER 3.

[4] CAYLEY 3; LAGUERRE 1; FROBENIUS 1, 2; WEYR 8; TABER 1; PASCH 1; BUCHHEIM 3; MOLIEN 1; SYLVESTER 1; SHAW 7; WHITEHEAD 1, and *Bibliography of Quaternions*.

428. Theorem. ϕ may be resolved according to the preceding theorem along any units of the form given by i, j, as

$$e'_{a'b'} = i'^{a'} j'^{b'}$$

If j be such that

$$S \,.\, j^s \phi = 0, j^n = 1 \qquad s = 1 \ldots . n - 1$$

then ϕ may be written in the form

$$\phi = x_{00} + \Sigma\, x_{ab}\, i^a j^b \qquad a = 1 \ldots . n - 1 \quad b = 0 \ldots . n - 1$$

429. Theorem. Whatever number β is, $\beta^t \phi \beta^{-t}$ has the same characteristic equation as ϕ. Hence if this equation is

$$\xi^n - m_1 \xi^{n-1} + m_2 \xi^{n-2} - \ldots . \pm m_n = 0$$

not only is ϕ a solution, but equally $\beta^t \phi \beta^{-t}$.

430. Definition. When

$$j^n = 1 \qquad S \,.\, j^s \phi = 0 \qquad s = 1 \ldots . n - 1$$

we shall call $j^t \phi j^{-t} = K^t \,.\, \phi$ the t-th conjugate of ϕ.

If ϕ is in the form of §428,

$$K^t \,.\, \phi = x_{00} + \Sigma\, x_{ab}\, \omega^{at}\, i^a j^b \qquad a = 1 \ldots . n - 1;\ b = 0 \ldots . n - 1$$

Hence $K^t \,.\, \phi$ is the same function[1] of $\omega^t i$ that ϕ is of i.

431. Theorem. We have at once

$$\phi + K \,.\, \phi + K^2 \,.\, \phi + \ldots . + K^{n-1} \,.\, \phi = n S \phi = m_1$$

$$(\phi + K \,.\, \phi + \ldots .)^2 = n^2 \,.\, S^2 \,.\, \phi = \Sigma \,.\, K^s \phi\, K^t \phi \qquad s, t = 0 \ldots . n - 1$$

and since

$$\phi^2 + (K\phi)^2 + \ldots . = \phi^2 + K \,.\, \phi^2 + \ldots . = n \,.\, S \,.\, \phi^2$$

therefore

$$2 \Sigma K^s \phi K^t \phi = n^2 S^2 \phi - n S \phi^2 = 2 m_2 \qquad s, t = 0 \ldots . n - 1, \qquad s \neq t$$

Similar equations may be deduced easily for $3 \,!\, m_3$, and the other coefficients.

432. Theorem. If $\phi \,.\, \sigma = g\sigma$, then

$$K^t \phi \,.\, j^t \sigma = g \,.\, j^t \sigma$$

also if

$$(\phi - g)\sigma_1 = \sigma_2 \ldots \ldots (\phi - g)^{u-1} \sigma_1 = \sigma_u \qquad (\phi - g)^u \sigma_1 = 0$$

then

$$(K^t \phi - g) j^t \sigma_1 = j^t \sigma_2 \ldots . (K^t \phi - g)^{u-1} j^t \sigma_1 = j^t \sigma_u \qquad (K^t \phi - g)^u j^t \sigma_1 = 0$$

433. Theorem. If the roots of ϕ are such that each latent factor $(\phi - g_i)$ occurs in the characteristic equation of ϕ to order unity only, then ϕ may be written

$$\phi = a_0 + a_1 i + \ldots . a_{n-1} i^{n-1}$$

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[1] Cf. TABER 2.

and

$$\phi \, . \, aj^s = (a_0 + a_1 + \ldots \; a_{n-1}) \, . \, aj^s$$
$$\phi \, . \, j^t aj^s = (a_0 + \omega^{-t} a_1 + \ldots \; \omega^{-t(n-1)} a_{n-1}) j^t aj^s$$

Hence the latent regions of $K^t\phi$ are simply those of ϕ transposed. This does not necessarily hold when the latent factors enter the characteristic equation to higher powers. We might equally say the roots of $K^t\phi$ are those of ϕ transposed (cyclically).

434. **Definition.** The transverse of ϕ with respect to the ground defined by i, j is

$$\breve{\phi} = \Sigma x_{ab} j^{-b} i^a = \Sigma x_{ab} \, \omega^{-ab} \, i^a j^{-b}$$

It is evident that $\breve{\breve{\phi}} = \phi$.

If $\breve{\phi}\phi = 1$ we call ϕ *orthogonal.* If $\phi = \breve{\phi}$ we call ϕ *symmetric* or *self-transverse.* If

$$\phi = \Sigma \, (x_{ij} + \sqrt{-1} \; y_{ij}) \, \lambda_{ij0}, \; (x, y \text{ real})$$

and if

$$\bar{\phi} = \Sigma \, (x_{ij} - \sqrt{-1} \; y_{ij}) \, \lambda_{ij0}$$

then ϕ is *real* if $\bar{\phi} = \phi$, *unitary* if $\bar{\breve{\phi}}\phi = 1$, *hermitian* if $\bar{\breve{\phi}} = \phi$.

435. **Theorem.** The transverse of $\phi\psi$ is $\breve{\psi}\breve{\phi}$. Consequently $\overbrace{\phi\breve{\phi}} = \phi\breve{\phi}$, and $\overbrace{\breve{\phi}\phi} = \breve{\phi}\phi$.

436. **Theorem.** We may write the equation of ϕ, if

$$\phi = a + bi + ci^2 + \ldots + ki^{n-1}$$

$$\begin{vmatrix} S\,.\,\phi - \phi & S\,.\,i\phi & S\,.\,i^2\phi \;\ldots & S\,.\,i^{n-1}\phi \\ S\,.\,i^{n-1}\phi & S\,.\,\phi - \phi & S\,.\,i\phi \;\ldots & S\,.\,i^{n-2}\phi \\ \ldots & \ldots & \ldots & \ldots \\ S\,.\,i\phi & S\,.\,i^2\phi & S\,.\,i^3\phi \;\ldots & S\,.\,\phi - \phi \end{vmatrix} = 0$$

So that

$$T_1\phi = nS\phi = \phi + K\phi + \ldots + K^{n-1}\phi$$
$$T_2\phi = \Sigma K^a\phi \, K^b\phi \qquad a, b = 0 \ldots n-1, \quad a \neq b$$
$$T_3\phi = \Sigma K^a\phi \, K^b\phi \, K^c\phi$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T_n\phi = \phi K\phi \, K^2\phi \ldots K^{n-1}\phi$$

It follows that if the characteristic function of ξ be formed, it may be written

$$(\xi - \phi) \, (\xi - K\phi) \ldots (\xi - K^{n-1}\phi)$$

By differentiating this expression *in situ* the characteristic function for $\xi_1 \ldots \xi_n$ may be formed in terms of $\phi_1 \ldots \phi_n$. This function will vanish for

$$\xi_1 = \phi_1 \ldots \xi_n = \phi_n$$

or for

$$\xi_1 = K^i\phi_1 \ldots \xi_n = K^i\phi_n \qquad (i = 1 \ldots n-1)$$

XX. PEIRCE ALGEBRAS.

437. In the following lists of algebras, the canonical notation explained above is used. In the author's opinion, it is the simplest method of expression. The subscripts only of the λ will be given; thus $(111) + a(122)$ means $\lambda_{111} + a\lambda_{122}$. For convenient reference the characteristic equation is given. The forms chosen as inequivalent are in many cases a matter of personal taste, but an attempt has been made to base the types upon the defining equations of the algebra. The designation of each algebra according to other writers[1] is given.

The only algebra of this type of order *one* is the idempotent unit

$$e_1 = \eta = \lambda_{110} = (110)$$

438. Order 2. *Type*[2] (η, i): $\qquad (x - x_2 e_2)^2 = 0$

$$e_2 = (110) \qquad e_1 = (111)$$

The product of $\zeta = x_1 e_1 + x_2 e_2$, $\sigma = y_1 e_1 + y_2 e_2$ is

$$\zeta\sigma = e_1(x_1 y_2 + x_2 y_1) + e_2(x_2 y_2)$$

The algebra may be defined in terms of any two numbers ζ, ζ^2, if $\zeta^2 \neq 0$, so that we may put σ in the form $\sigma = x\zeta + y\zeta^2$.

439. Order 3. *Type*[3] (η, i, i^2): $\qquad (x - x_3 e_3)^3 = 0$

$$e_3 = (110) \qquad e_2 = (111) \qquad e_1 = (112)$$

The general product is

$$\zeta\sigma = e_1(x_1y_3 + x_2y_2 + x_3y_1) + e_2(x_2y_3 + x_3y_2) + e_3(x_3y_3)$$

The algebra may be defined in terms of ζ, ζ^2, ζ^3, if $\zeta^2 \neq 0$, $\zeta^3 \neq 0$.

Type[4] (η, i, j): $\qquad (x - x_3 e_3)^2 = 0$

$$e_3 = (110) + (220) \qquad e_2 = (210) \qquad e_1 = (111)$$

$$\zeta\sigma = e_1(x_1 y_3 + x_3 y_1) + e_2(x_2 y_3 + x_3 y_2) + e_3 x_3 y_3 = \sigma\zeta$$

The algebra is definable by any two numbers ζ, σ whose product does not vanish. The product of $\zeta\sigma$ may be written

$$\zeta\sigma = \sigma S\zeta + \zeta S\sigma - e_3 S\zeta S\sigma$$

Hence

$$e_3 S\zeta S\sigma = \sigma S\zeta + \zeta S\sigma - \zeta\sigma$$

Also we may write the algebra (η, ζ', σ'), where ζ', σ' are nilpotents, $\zeta'\sigma' = 0 = \sigma'\zeta'$.

440. Order 4. *Type*[5] (η, i, i^2, i^3): $\qquad (x - x_4 e_4)^4 = 0$

$$e_4 = (110) \qquad e_3 = (111) \qquad e_2 = (112) \qquad e_1 = (113)$$

If $\zeta = S\zeta + V\zeta$, then the algebra is defined by

$$\zeta,\ \zeta^2,\ \zeta^3,\ \zeta^4, \text{ if } S\zeta \neq 0,\ \ V\zeta \neq 0,\ \ (V\zeta)^2 \neq 0,\ \ (V\zeta)^3 \neq 0$$

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[1] Enumerations are given by PINCHERLE 1; CAYLEY 8; STUDY 1, 2, 3, 8; SCHEFFERS 1, 2, 3; PEIRCE 3; ROHR 1; STARKWEATHER 1, 2; HAWKES 1, 3, 4.

[2] STUDY II; SCHEFFERS II_1; PEIRCE a_2.

[3] STUDY III; SCHEFFERS III_1; PEIRCE a_3.

[4] STUDY V; SCHEFFERS III_3.

[5] STUDY V; SCHEFFERS IV_1; PEIRCE a_4.

Type (η, i, j, j^2): $\qquad (x - x_4 e_4)^3 = 0$

$$e_4 = (110) + (220) \qquad e_3 = (210) + a\,(122) \qquad e_2 = (111) + b\,(122) \qquad e_1 = (112)$$

$$\zeta\sigma = -S\zeta \,.\, S\sigma + \sigma S\zeta + \zeta S\sigma + e_1 (x_2 y_2 + a\, x_3 y_3 + b\, x_2 y_3)$$

or

$$V\zeta \,.\, V\sigma = e_1 (x_2 y_2 + a\, x_3 y_3 + b\, x_2 y_3)$$

Hence

$$V\zeta \,.\, V\sigma - V\sigma \,.\, V\zeta = \zeta\sigma - \sigma\zeta = b e_1 (x_2 y_3 - x_3 y_2)$$

We have two cases then: (1) when $b = 0$, (2) when $b \neq 0$. We may determine $e_2^2 = e_1$, from

$$(V\zeta)^2 = e_1 (x_2^2 + a x_3^2)$$

When $a = 0$, this gives us only one case of $\sigma^2 = e_1$.
When $a \neq 0$, we may take $e_3^2 = e_1$ as well as $e_2^2 = e_1$; whence, if $a = 0$

$$e_2 e_3 = 0 \qquad\qquad e_3 e_2 = 0$$

If $a \neq 0$, we may put $a = 1$

$$e_3 e_2 = 0 \qquad\qquad e_2 e_3 = 0$$

Finally, then, we have[1]

$(\eta\, i j j^2)\,(1)$	$e_3 = (210)$	$e_2 = (111)$	$e_1 = (112)$
$(\eta\, i j j^2)\,(2)$	$e_3 = (210) + (122)$	$e_2 = (111)$	$e_1 = (112)$
$(\eta\, i j j^2)\,(3)$	$e_3 = (210) + (122)\, a$	$e_2 = (111) + (122)$	$e_1 = (112)$
$(\eta\, i j j^2)\,(4)$	$e_3 = (210)$	$e_2 = (111) + (122)$	$e_1 = (112)$

Type (η, i, j, ij): $\qquad (x - x_4 e_4)^2 = 0$

$$e_4 = (110) + (220) \qquad e_3 = (210) \qquad e_2 = (111) - (221) \qquad e_1 = (211)$$

$$\zeta\sigma = e_1 (x_3 y_2 - x_2 y_3 + x_3 y_4 + x_4 y_3) + e_2 (x_2 y_4 + x_4 y_2) + e_3 (x_3 y_4 + x_4 y_3) + e_4 x_4 y_4$$

Defined[2] by ζ, σ, such that $(V\zeta)^2 = 0 = (V\sigma)^2$

$$V\zeta \,.\, V\sigma = -V\sigma V\zeta$$

Type[3] (η, i, j, k): $\qquad (x - x_4 e_4)^2 = 0$

$$e_4 = (110) + (220) + (330) \qquad e_3 = (210) \qquad e_2 = (310) \qquad e_1 = (111)$$

$$V\zeta\, V\sigma = 0$$

Defined by any three independent numbers.

441. Order 5. *Type*[4] (η, i, i^2, i^3, i^4): $\qquad (x - x_5 e_5)^5 = 0$

$$e_5 = (110) \qquad e_4 = (111) \qquad e_3 = (112) \qquad e_2 = (113) \qquad e_1 = (114)$$

Definable by any number ζ for which $(V\zeta)^4 \neq 0$.

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[1] STUDY IX is (η, i, j, j^2) (3) if $e'_3 = (210) - (111) + (c-1)(122)$, $e_2 = (111) + 2(122)$. SCHEFFERS IV_3 is the same. PEIRCE b_4 and b'_4 reduce to this form. STUDY X and SCHEFFERS IV_4 reduce to (2); STUDY XI and SCHEFFERS IV_5 reduce to (1); SCHEFFERS IV_6 reduces to (4) if $\lambda = -1$, otherwise it reduces to (3).

[2] STUDY XIV; SCHEFFERS IV_8; PEIRCE d_4.

[3] STUDY XVI; SCHEFFERS IV_9.

[4] SCHEFFERS V_1; PEIRCE a_5.

Type[1] (η, i, j, j^2, j^3): $\quad (x - x_5 e_5)^4 = 0$

$e_5 = (110) + (220) \qquad e_4 = (210) + a(123)$

$e_3 = (111) + b(123) \qquad e_2 = (112) \qquad e_1 = (113)$

(1) $b \neq 0$, we may take $b = 1$.
(2) $b = 0$, we may take $a = 1$, or
(3) $b = 0 = a$.

Type[2] (η, i, j, ij, j^2): $\quad (x - x_5 e_5)^3 = 0$

$e_5 = (110) + (220) \qquad e_4 = (210) + b(221) + c(122)$

$e_3 = (111) + d(221) + e(122) \qquad e_2 = (211) \qquad e_1 = (112)$

(1) $e_4 = (210) \qquad e_3 = (111) + d(221)$
(2) $e_4 = (210) \qquad e_3 = (111) + d(221) + (122)$
(3) $e_4 = (210) + (122) \qquad e_3 = (111)$
(4) $e_4 = (210) + (221) \qquad e_3 = (111) - (221) + e(122)$
(5) $e_4 = (210) + (122) \qquad e_3 = (111) + d(221) + e(122)$

Type[3] (η, i, i^2, j, j^2): $\quad (x - x_5 e_5)^3 = 0$

$e_5 = (110) + (220) + (330) \qquad e_4 = (210) + (320)$

$e_3 = (310) \qquad e_2 = (111) \qquad e_1 = (112)$

Type[4] (η, i, j, k, k^2): $\quad (x - x_5 e_5)^3 = 0$

$e_5 = (110) + (220) + (330) \qquad e_4 = (210) + a(122) + b(132)$

$e_3 = (310) + c(122) + d(132) \qquad e_2 = (111) + e(122) + f(132) \qquad e_1 = (112)$

(1) $e_4 = (210) + (122) \qquad e_3 = (310) + (132) \qquad e_2 = (111) \qquad e_1 = (112)$
(2) $e_4 = (210) \qquad e_3 = (310) + (132)$ . . . .
(3) $e_4 = (210) \qquad e_3 = (310)$ . . . .
(4) $e_4 = (210) + (122) - \gamma(132) \qquad e_3 = (310) + \gamma(122) - (132) \qquad e_2 = (111)$
(5) $e_4 = (210) + (122) - (132) \qquad e_3 = (310) + (122)$ . . . .
(6) $e_4 = (210) - (132) \qquad e_3 = (310) + (132)$ . . . .
(7) $e_4 = (210) + (1 + \lambda^2)(122) \qquad e_3 = (310)$ . . . .
(8) $e_4 = (210) + (122) \qquad e_3 = (310) \qquad e_2 = (111) - 2(122)$ . . . .
(9) $e_4 = (210) + (122) \qquad e_3 = (310) + 2(122) - (132)$
$\qquad e_2 = (111) - 2(122)$
(10) $e_4 = (210) - (122) + (132) \qquad e_3 = (310) - i(122) - (132)$
$\qquad e_2 = (111) - 2i(132)$

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[1] Scheffers V_4 is in (1), $e_4 = (210) + (123) - (112)$, $e_3 = (111) + 2(123)$; Scheffers V_5 is (2); Scheffers V_6 is in case (1), $a = 0$, $e_4 = (210) - (112)$; Scheffers V_7 is (3); Peirce b_5 is in (1), $j = (111) - (123)$, $k = (112)$, $l = (113)$, $m = (210) + (123) + (112)$; Peirce c_5 is in (1), $j = (111) - (123)$, $k = (112)$, $l = (113)$, $m = (210) + (112)$.

[2] Scheffers V_{15} is (1); $e_4 = (111) + \lambda(221)$, $e_3 = (210)$; V_{17} is (2) with $d = -1$; V_{18} is in (5); V_{19} is in (2) or (4); Peirce d_5 is in (5); e_5 is in (4); f_5 is in (1); g_5 is in (5); h_5 is in (3); i_5 is in (1).

[3] Scheffers V_{16}; Peirce j_5.

[4] These are in order Scheffers $V_{20} - V_{29}$.

Type (η, i, j, k, l): $(x - x_5 e_5)^2 = 0$

(1)[1] $e_5 = (110) + (220) + (330) + (440)$ $e_4 = (210) - (131)$
$e_3 = (310) + (121)$ $e_2 = (410)$ $e_1 = (111)$

(2)[2] $e_5 = (110) + (220) + (330) + (440)$ $e_4 = (210)$
$e_3 = (310)$ $e_2 = (410)$ $e_1 = (111)$

442. Order 6. *Type*[3] $(\eta, i, i^2, i^3, i^4, i^5)$: $(x - x_6 e_6)^6 = 0$

$e_6 = (110)$ $e_5 = (111)$ $e_4 = (112)$ $e_3 = (113)$ $e_2 = (114)$ $e_1 = (115)$

Type[4] $(\eta, i, j, j^2, j^3, j^4)$: $(x - x_6 e_5)^5 = 0$

$e_6 = (110) + (220)$ $e_5 = (210) + a(124)$ $e_4 = (111) + b(124)$
$e_3 = (112)$ $e_2 = (113)$ $e_1 = (114)$

(1) $a = 1 = b$ $e_5 = (210) + (124)$ $e_4 = (111) + (124)$
(2) $a = 0, b = 1$ $e_5 = (210)$ $e_4 = (111) + (124)$
(3) $a = 0 = b$ $e_5 = (210)$ $e_4 = (111)$

Type[5] $(\eta, i, j, ij, j^2, j^3)$ $(x - x_6 e_6)^4 = 0$

(1) $e_5 = (210) + (122) + 2\sqrt{-1}(221)$ $e_4 = (111) + (221)$
(2) $e_5 = (210)$ $e_4 = (111) + 2(123)$
(3) $e_5 = (210) + (123)$ $e_4 = (111) + 2(123)$
(4) $e_5 = (210)$ $e_4 = (111) + d(221)$
(5) $e_5 = (210) + (221)$ $e_4 = (111)$
(6) $e_5 = (210) + (123)$ $e_4 = (111) + d(221)$
(7) $e_5 = (210) + (221)$ $e_4 = (111) + (123)$
(8) $e_5 = (210)$ $e_4 = (111)$
(9) $e_5 = (210) + (123)$ $e_4 = (111)$
(10) $e_5 = (210) + (122)$ $e_4 = (111) - (221) - 2(122)$
(11) $e_5 = (210) + (122)$ $e_4 = (111) - (221)$
(12) $e_5 = (210) + (123)$ $e_4 = (111) - (221) - 2(122)$
(13) $e_5 = (210)$ $e_4 = (111) - (221) - 2(122)$
(14) $e_5 = (210) + 2(1 \mp \sqrt{-1})(221) + 4\sqrt{-1}(122) + (123)$
$e_4 = (111) \mp \sqrt{-1}(221) + 2(1 \pm \sqrt{-1})122$
(15) $e_5 = (210) + 2\sqrt{-1}(221) + (122)$ $e_4 = (111) + (221) + 2(123)$
(16) $e_5 = (210) + 4(221) + (123)$ $e_4 = (111) + (221) + 2(122)$
(17) $e_5 = (210) + 4(221)$ $e_4 = (111) + (221) + 2(122)$
(18) $e_5 = (210) + 4(221) + (123)$ $e_4 = (111) + 4(122)$
(19) $e_5 = (210) - (m-1)(221) - \frac{1}{4}(m+1)(m-3)(122)$
$$e_4 = (111) + \frac{m+1}{m-3}(221) + 2(122)$$

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[1] SCHEFFERS V_{32}. [2] SCHEFFERS V_{33}. [3] PEIRCE a_6. [4] PEIRCE b_6 is (1); c_6 is (2).

[5] These are in order STARKWEATHER 4, 8, 9, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 27, 29, 30, 32, 33. Also PEIRCE aa_6 and w_6 are in (4), ad_6 in (5), z_6 in (6), af_6 in (8), ae_6 in (9), u_6 in (11).

Type[1] $(\eta, i, i^2, j, j^2, j^3)$ $(x - x_6 e_6)^4 = 0$

(1) $e_5 = (210) + (320)$ $e_4 = (310)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(2) $e_5 = (210) + (320) + (133)$ $e_4 = (310) + (123)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(3) $e_5 = (210) + (320) + (133)$ $e_4 = (310) + (123)$ $e_3 = (111) + 2\,(123)$ $e_2 = (112)$ $e_1 = (113)$

(4) $e_5 = (210) + (320)$ $e_4 = (310)$ $e_3 = (111) + 2\,(123)$ $e_2 = (112)$ $e_1 = (113)$

Type[2] $(\eta, i, j, k, k^2, k^3)$ $(x - x_6 e_6)^4 = 0$

(1) $e_5 = (210)$ $e_4 = (310)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(2) $e_5 = (210) + (123)$ $e_4 = (310)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(3) $e_5 = (210)$ $e_4 = (310)$ $e_3 = (111) + 2\,(123)$ $e_2 = (112)$ $e_1 = (113)$

(4) $e_5 = (210) - (133)$ $e_4 = (310) + (123)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(5) $e_5 = (210)$ $e_4 = (310)$ $e_3 = (111) + 2\,(133)$ $e_2 = (112)$ $e_1 = (113)$

(6) $e_5 = (210) + g\,(133)$ $e_4 = (310) + (123)$ $e_3 = (111)$ $e_2 = (112)$ $e_1 = (113)$

(7) $e_5 = (210) + (133)$ $e_4 = (310) + (123)$ $e_3 = (111) + 2\,(123)$ $e_2 = (112)$ $e_1 = (113)$

(8) $e_5 = (210) + (133) + (123)$ $e_4 = (310) + (123)$ $e_3 = (111) + 2\,(133)$ $e_2 = (112)$ $e_1 = (113)$

Type $(\eta, i, j, ij, j^2, ij^2)$ $\omega^3 = 1$ $(x - x_6 e_6)^3 = 0$

$e_5 = (210) + \frac{1}{2}(1 - \omega)(221) - \frac{3}{4}\omega(122)$ $e_4 = (111) + \omega(221) - \frac{1}{2}(1 - \omega)(122)$

$e_3 = (211) + \frac{1}{4}(1 - \omega)(222)$ $e_2 = (112) + \omega^2(222)$ $e_1 = (212)$

Type (η, i, j, k, ik, k^2) $(x - x_6 e_6)^3 = 0$

(1) $e_5 = (210)$ $e_4 = (310) + (132)$ $e_3 = (111) + b\,(122) + c\,(132)$ $e_2 = (211)$ $e_1 = (112)$

(2) $e_5 = (210) + (122)$ $e_4 = (310) + (132)$ $e_3 = (111) + b\,(122) + c\,(132)$ $e_2 = (211)$ $e_1 = (112)$

(3) $e_5 = (210) + a\,(122)$ $e_4 = (310) + (122) + (132)$ $e_3 = (111) + b\,(122) + c\,(132)$ $e_2 = (211)$ $e_1 = (112)$

(4) $e_5 = (210)$ $e_4 = (310) + (122) + (132)$ $e_3 = (111) + b\,(122) + c\,(132)$ $e_2 = (211)$ $e_1 = (112)$

(5) $e_5 = (210) + (132)$ $e_4 = (310) - (122)$ $e_3 = (111)$ $e_2 = (211)$

(6) $e_5 = (210)$ $e_4 = (310)$ $e_3 = (111) + (122)$ $e_2 = (211)$ $e_1 = (112)$

(7) $e_5 = (210)$ $e_4 = (310)$ $e_3 = (111)$ $e_3 = (211)$ $e_1 = (112)$

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[1] These are in order STARKWEATHER 3, 5, 28, 10.

[2] These are in order STARKWEATHER 1, 2, 6, 17, 18, 24, 25, 26.

Type (η, i, j, k, l, l^2) $\qquad (x - x_6 e_6)^3 = 0$

(1)	$e_5 = (210) - a(132)$	$e_4 = (310) + a(122)$	$e_3 = (410)$
		$e_2 = (111) + a(122) + a(132) + a(142)$	$e_1 = (112)$
(2)		$e_2 = (111) + a(122) + a(132)$	
(3)		$e_2 = (111) + a(122) + a(142)$	
(4)		$e_2 = (111) + a(132)$	
(5)		$e_2 = (111) + a(142)$	
(6)		$e_2 = (111)$	
(7)	$e_5 = (210)$	$e_4 = (310)$	$e_3 = (410)$
		$e_2 = (111) + (122) + (132) + (142)$	$e_1 = (112)$
(8)		$e_2 = (111) + (122) + (132)$	
(9)		$e_2 = (111) + (122)$	
(10)		$e_2 = (111)$	

Type (η, i, j, k, l, il) $\qquad (x - x_6 e_6)^2 = 0$

(1)	$e_5 = (210) - (231)$	$e_4 = (310) + (221)$	$e_3 = (410)$
		$e_2 = (111) + (221) + (231) + (241)$	$e_1 = (211)$
(2)		$e_2 = (111) + (221) + (231)$	
(3)		$e_2 = (111) + (221) + (241)$	
(4)		$e_2 = (111) + (221)$	
(5)		$e_2 = (111) + (241)$	
(6)		$e_2 = (111)$	
(7)	$e_5 = (210)$	$e_4 = (310)$	$e_3 = (410)$
		$e_2 = (111) + (221) + (231) + (241)$	$e_1 = (211)$
(8)		$e_2 = (111) + (221) + (231)$	
(9)		$e_2 = (111) + (221)$	
(10)		$e_2 = (111)$	

Type (η, i, j, k, l, m) $\qquad (x - x_6 e_6)^2 = 0$

$e_5 = (210)$	$e_4 = (310)$	$e_3 = (410)$	$e_2 = (510)$	$e_1 = (111)$

XXI. SCHEFFERS ALGEBRAS.

443. The following lists include algebras of order less than seven, with more than one idempotent. Reducible algebras are not included, nor are reciprocal algebras both given.[1] The idempotents are η; direct units $i, j \ldots$; skew units e.

444. Order 3. *Type*[2] $(\eta_1;\ \eta_2;\ e_{21})$ $\qquad (x - x_2 e_0)(x - x_3 e_0) = 0$

$e_3 = (110) \qquad e_2 = (220) \qquad e_1 = (210)$

445. Order 4. *Type*[3] $(\eta_1;\ \eta_2,\ i;\ e_{12})$ $\qquad (x - x_3 e_0)(x - x_4 e_0)^2 = 0$

$e_4 = (220) \qquad e_3 = (110) \qquad e_2 = (111) \qquad e_1 = (210)$

Type[4] $(\eta_1;\ \eta_2;\ e_{21},\ e'_{21})$ $\qquad (x - x_3 e_0)(x - x_4 e_0) = 0$

$e_4 = (110) \qquad e_3 = (220) \qquad e_2 = (210) \qquad e_1 = (211)$

Type[5] $(\eta_1;\ \eta_2;\ e_{21},\ e_{12})$ $\qquad (x - x_3 e_0)(x - x_4 e_0) = 0$

$e_4 = (110) \qquad e_3 = (220) \qquad e_2 = (121) \qquad e_1 = (211)$

446. Order 5. *Type*[6] $(\eta_1,\ i,\ i^2;\ \eta_2,\ e_{21})$ $\qquad (x - x_4 e_0)(x - x_5 e_0)^3 = 0$

$e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (111) \qquad e_2 = (112) \qquad e_1 = (211)$

Type[7] $(\eta_1,\ i;\ \eta_2,\ j;\ e_{21})$ $\qquad (x - x_4 e_0)^2 (x - x_5 e_0)^2 = 0$

$e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (111) \qquad e_2 = (222) \qquad e_1 = (211)$

Type[8] $(\eta_1,\ i;\ \eta_2;\ e_{21},\ e'_{21})$ $\qquad (x - x_4 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (111) \qquad e_2 = (211) \qquad e_1 = (212)$

(2) $e_5 = (110) \qquad e_4 = (220) + (330) \qquad e_3 = (111) \qquad e_2 = (211) \qquad e_1 = (310)$

(3) $e_5 = (110) + (220) \qquad e_4 = (330) \qquad e_3 = (210) \qquad e_2 = (111) \qquad e_1 = (311)$

Type[9] $(\eta_1,\ i;\ \eta_2;\ e_{12},\ e_{21})$ $\qquad (x - x_4 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (122) \qquad e_2 = (210) \qquad e_1 = (112)$

(2) $e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (122) \qquad e_2 = (211) \qquad e_1 = (112)$

Type[10] $(\eta_1;\ \eta_2;\ e'_{21},\ e''_{21},\ e'''_{21})$ $\qquad (x - x_4 e_0)(x - x_5 e_0) = 0$

$e_5 = (110) \qquad e_4 = (220) + (330) + (440) \qquad e_3 = (211) \qquad e_2 = (310) \qquad e_1 = (410)$

Type[11] $(\eta_1;\ \eta_2;\ e_{12},\ e'_{21},\ e''_{21})$ $\qquad (x - x_4 e_0)(x - x_5 e_0) = 0$

$e_5 = (110) \qquad e_4 = (220) + (330) \qquad e_3 = (121) \qquad e_2 = (211) \qquad e_1 = (310)$

Type[12] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{21},\ e_{31})$ $\qquad (x - x_3 e_0)(x - x_4 e_0)(x - x_5 e_0) = 0$

$e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (330) \qquad e_2 = (221) \qquad e_1 = (311)$

Type[13] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{21},\ e_{32})$ $\qquad (x - x_3 e_0)(x - x_4 e_0)(x - x_5 e_0) = 0$

$e_5 = (110) \qquad e_4 = (220) \qquad e_3 = (330) \qquad e_2 = (211) \qquad e_1 = (321)$

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[1] For algebras of order seven see Hawkes 4.

[2] Study IV; Scheffers III_2.

[3] Study VII; Scheffers IV_2.

[4] Study XV; Scheffers IV_6.

[5] Study XIII; Scheffers IV_7.

[6] Scheffers V_2; Hawkes (V) 1_1.

[7] Scheffers V_3; Hawkes (V) 4.

[8] These are in order Scheffers V_{10}, V_{11}, V_{14}; Hawkes (V) 2_{11}, 2_{12}, 1_2.

[9] These are in order Scheffers V_{12}, V_{13}; Hawkes (V) 3_1, 3_2.

[10] Scheffers V_{20}; Hawkes (V) 5.

[11] Scheffers V_{21}; Hawkes (V) 6.

[12] Scheffers V_8.

[13] Scheffers V_9.

447. Order 6. *Type*[1] $(\eta_1,\ i,\ i^2,\ i^3;\ \eta_2;\ e_{21})$ $\qquad (x - x_5 e_0)(x - x_6 e_0)^4 = 0$

$e_6 = (110) \quad e_5 = (220) \quad e_4 = (221) \quad e_3 = (222) \quad e_2 = (223) \quad e_1 = (210)$

Type[2] $(\eta_1,\ i_1,\ j_1,\ j_1^2;\ \eta_2;\ e_{21})$ $\qquad (x - x_5 e_0)(x - x_6 e_0)^3 = 0$

(1) $e_6 = (220) + (330) \quad e_5 = (110) \quad e_4 = (320) \quad e_3 = (221) \quad e_2 = (222)$
$e_1 = (212)$

(2) $e_6 = (220) + (330) \quad e_5 = (110) \quad e_4 = (320) + (232) \quad e_3 = (221)$
$e_2 = (222) \quad e_1 = (212)$

(3) $e_6 = (220) + (330) \quad e_5 = (110) \quad e_4 = (320) + a(232)$
$e_3 = (221) + (232) \quad e_2 = (222) \quad e_1 = (212)$

(4) $e_6 = (220) + (330) \quad e_5 = (110) \quad e_4 = (320) \quad e_3 = (221) + (232)$
$e_2 = (222) \quad e_1 = (212)$

Type[3] $(\eta_1,\ i_1,\ j_1,\ i_1 j_1;\ \eta_2;\ e_{21})$ $\qquad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

$e_6 = (110) \quad e_5 = (220) + (330) \quad e_4 = (221) - (331) \quad e_3 = (320) \quad e_2 = (321)$
$e_1 = (211)$

Type[4] $(\eta_1,\ i_1,\ j_1,\ k_1;\ \eta_2;\ e_{21})$ $\qquad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

$e_6 = (110) \quad e_5 = (220) + (330) + (440) \quad e_4 = (320) \quad e_3 = (420) \quad e_2 = (221)$
$e_1 = (211)$

Type[5] $(\eta_1,\ i_1,\ i_1^2;\ \eta_2,\ i_2;\ e_{21})$ $\qquad (x - x_5 e_0)^2 (x - x_6 e_0)^3 = 0$

$e_6 = (220) \quad e_5 = (110) \quad e_4 = (221) \quad e_3 = (111) \quad e_2 = (112), \quad e_1 = (122)$

Type[6] $(\eta_1,\ i_1,\ j_1;\ \eta_2,\ i_2;\ e_{21})$ $\qquad (x - x_5 e_0)^2 (x - x_6 e_0)^2 = 0$

$e_6 = (330) \quad e_5 = (110) + (220) \quad e_4 = (331) \quad e_3 = (210) \quad e_2 = (111) \quad e_1 = (311)$

Type[7] $(\eta,\ i_1,\ i_1^2;\ \eta_2;\ e_{12},\ e_{12}')$ $\qquad (x - x_5 e_0)(x - x_6 e_0)^3 = 0$

(1) $e_6 = (110) \quad e_5 = (220) + (330) + (440) \quad e_4 = (221) + (430) \quad e_3 = (222)$
$e_2 = (310) \quad e_1 = (410)$

(2) $e_6 = (110) \quad e_5 = (220) + (330) + (440) \quad e_4 = (221) \quad e_3 = (222)$
$e_2 = (310) \quad e_1 = (410)$

Type[8] $(\eta_1,\ i_1,\ i_1^2;\ \eta_2;\ e_{12},\ e_{21})$ $\qquad (x - x_6 e_0)(x - x_5 e_0)^3 = 0$

(1) $e_6 = (330) + (440) \quad e_5 = (110) + (220) \quad e_4 = (132) \quad e_3 = (310) \quad e_2 = (111)$
$e_1 = (112)$

(2) .... $e_4 = (142)$ ....

Type[9] $(\eta_1,\ i_1,\ j_1;\ \eta_2;\ e_{21},\ e_{21}')$ $\qquad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_6 = (110) \quad e_5 = (220) + (330) + (440) + (550) \quad e_4 = (320) + (540)$
$e_3 = (221) \quad e_2 = (410) \quad e_1 = (510)$

(2) .... $e_4 = (320)$ ....

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[1] HAWKES (VI) 1_4 1.
[2] In order HAWKES (VI) 1_4 3, 1_4 4, 1_4 2, ——.
[3] HAWKES (VI) 1_4 5.
[4] HAWKES (VI) 1_4 6.
[5] HAWKES (VI) 2_4 1.
[6] HAWKES (VI) 2_4 2.
[7] HAWKES (VI) 3_4 1, 3_4 2.
[8] HAWKES (VI) 4_4 1, 4_4 3.
[9] HAWKES (VI) 3_4 3; 3_4 4.

Type[1] $(\eta_1, i_1, j_1;\ \eta_2;\ e_{21}, e_{12})$ $\quad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_6 = (330) + (440) \quad e_5 = (110) + (220) + (550) \quad e_4 = (131) \quad e_3 = (310)$
$e_2 = (210) \quad e_1 = (111)$

(2) . . . . $e_4 = (141) \quad e_3 = (311)$ . . . .

Type[2] $(\eta_1, i_1;\ \eta_2, i_2;\ e_{12}, e'_{12})$ $\quad (x - x_6 e_0)^2 (x - x_5 e_0)^2 = 0$

(1) $e_6 = (440) \quad e_5 = (110 + (220) + (330) \quad e_4 = (111) \quad e_3 = (441) \quad e_2 = (140)$
$e_1 = (141)$

(2) $e_6 = (440) \quad e_5 = (110) + (220) \quad e_4 = (111) \quad e_3 = (441) \quad e_2 = (240)$
$e_1 = (241)$

(3) $e_6 = (440) \quad e_5 = (110) + (220) + (330) \quad e_4 = (111) \quad e_3 = (441) \quad e_2 = (340)$
$e_1 = (240)$

Type[3] $(\eta_1, i_1;\ \eta_2, i_2;\ e_{12}, e_{21})$ $\quad (x - x_6 e_0)^2 (x - x_5 e_0)^2 = 0$

(1) $e_6 = (330) + (440) \quad e_5 = (110) + (220) \quad e_4 = (310) + (421)$
$e_3 = (131) + (240) \quad e_2 = (441) \quad e_1 = (111)$

(2) . . . . $e_3 = (240)$ . . . .

(3) . . . . $e_4 = (310) \quad e_3 = (131) + (240)$ . . . .

(4) . . . . $e_3 = (240)$ . . . .

Type[4] $(\eta_1, i_1;\ \eta_2;\ e'_{12}, e''_{12}, e'''_{12})$ $\quad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_6 = (440) \quad e_5 = (110) + (220) + (330) \quad e_4 = (111) \quad e_3 = (340)$
$e_2 = (140) \quad e_1 = (141)$

(2) $e_6 = (550) \quad e_5 = (110) + (220) + (330) + (440) \quad e_4 = (111) \quad e_3 = (150)$
$e_2 = (250) \quad e_1 = (151)$

(3) . . . . $e_3 = (350)$
. . . . $e_1 = (450)$

Type[5] $(\eta_1, i_1;\ \eta_2;\ e'_{12}, e''_{12}, e_{21})$ $\quad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

(1) $e_6 = (440) + (550) \quad e_5 = (110) + (220) + (330) \quad e_4 = (530) \quad e_3 = (140)$
$e_2 = (111) \quad e_1 = (141)$

(2) . . . . $e_3 = (141)$
. . . . $e_1 = (240)$

Type[6] $(\eta_1, i_1;\ \eta_2;\ e'_{21}, e''_{21}, e_{12})$ $\quad (x - x_6 e_0)(x - x_5 e_0)^2 = 0$

$e_6 = (440) + (550) \quad e_5 = (110) + (220) + 330) \quad e_4 = (410) \quad e_3 = (141)$
$e_2 = (111) \quad e_1 = (530)$

Type[7] $(\eta_1;\ \eta_2;\ e'_{12}, e''_{12}, e'''_{12}, e^{\text{IV}}_{12})$ $\quad (x - x_6 e_0)(x - x_5 e_0) = 0$

$e_6 = (660) \quad e_5 = (110) + (220) + (330) + (440) + (550) \quad e_4 = (460) \quad e_3 = (360)$
$e_2 = (260) \quad e_1 = (160)$

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[1] HAWKES 4_4 2, 4_4 4.
[2] HAWKES (VI) 5_4 1, 5_4 3, 5_4 4.
[3] HAWKES (VI) 6_4 1, 6_4 2, 6_4 3, 6_4 4.
[4] HAWKES (VI) 5_4 2, 7_4 1, 7_4 2.
[5] HAWKES (VI) 8_4 1, 8_4 3.
[6] HAWKES (VI) 8_4 2.
[7] HAWKES (VI) 9_4.

Type[1] $(\eta_1;\ \eta_2;\ e'_{12},\ e''_{12},\ e'''_{12},\ e_{21})$ $\qquad (x - x_6 e_0)(x - x_5 e_0) = 0$

$e_6 = (110) + (220) + (330) + (440)\quad e_5 = (550) + (660)\quad e_4 = (460)\quad e_3 = (360)$

$e_2 = (260)\quad e_1 = (510)$

Type[2] $(\eta_1;\ \eta_2;\ e'_{12}\, e''_{12}\, e'_{21}\, e''_{21})$ $\qquad (x - x_6 e_0)\,(x - x_5 e_0) = 0$

$e_6 = (440) + (550) + (660)\quad e_5 = (110) + (220) + (330)\quad e_4 = (630)$

$e_3 = (530)\quad e_2 = (250)\quad e_1 = (140)$

Type[3] $(\eta_1;\ i_1;\ \eta_2;\ \eta_3;\ e_{12},\ e_{13})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0)^2 = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (313)\quad e_2 = (323)\quad e_1 = (333)$

Type[4] $(\eta_1,\ i_1;\ \eta_2;\ \eta_3;\ e_{21},\ e_{23})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0)^2 = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (212)\quad e_2 = (232)\quad e_1 = (331)$

Type[5] $(\eta_1,\ i_1;\ \eta_2;\ \eta_3;\ e_{13},\ e_{21})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0)^2 = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (231)\quad e_2 = (312)\quad e_1 = (332)$

Type[6] $(\eta_1,\ i_1;\ \eta_2;\ \eta_3;\ e_{21},\ e_{32})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0)^2 = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (121)\quad e_2 = (231)\quad e_1 = (331)$

Type[7] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{12},\ e_{13},\ e_{23})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0) = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330) + (440)\quad e_3 = (311)\quad e_2 = (420)\quad e_1 = (321)$

Type[8] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{12},\ e_{13},\ e_{32})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0) = 0$

(1) $e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (312)\quad e_2 = (231)\quad e_1 = (322)$

(2) . . . . $e_3 = (211)\quad e_2 = (320)\quad e_1 = (311)$

(3) . . . . $e_2 = (321)\quad e_1 = (311)$

Type[9] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{12},\ e'_{12},\ e_{31})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0) = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330) + (440)\quad e_3 = (420)\quad e_2 = (130)\quad e_1 = (321)$

Type[10] $(\eta_1;\ \eta_2;\ \eta_3;\ e_{12},\ e_{23},\ e_{31})$ $\qquad (x - x_4 e_0)\,(x - x_5 e_0)\,(x - x_6 e_0) = 0$

$e_6 = (110)\quad e_5 = (220)\quad e_4 = (330)\quad e_3 = (211)\quad e_2 = (131)\quad e_1 = (321)$

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[1] HAWKES (VI) 10_4.
[2] HAWKES (VI) 11_4.
[3] HAWKES (VI) 1_3 2.
[4] HAWKES (VI) 6_3.
[5] HAWKES (VI) 2_3 2.
[6] HAWKES (VI) 7_3.
[7] HAWKES (VI) 3_3.
[8] HAWKES (VI) 4_3, 9_3 1, 9_3 2.
[9] HAWKES (VI) 5_3.
[10] HAWKES (VI) 8_3.

XXII. CARTAN ALGEBRAS.

448. Quadrates. The units in this case have been given.

Dedekind Algebras. These have been considered.

Order[1] 7. $e_1 = (110)$ $e_2 = (120)$ $e_3 = (210)$ $e_4 = (220)$
$e_5 = (330)$ $e_6 = (130)$ $e_7 = (230)$

Order 8. *Type* $Q_1 \times (\eta, i)$

$$\begin{vmatrix} x_1 e_0 - x & x_2 \\ x_3 & x_4 e_0 - x \end{vmatrix}^2 = 0$$

This is biquaternions.

Type[2] $Q_1 + (\eta, i) + e_{23}$

$$\begin{vmatrix} x_1 e_0 - x & x_2 \\ x_3 & x_4 e_0 - x \end{vmatrix} (x_5 e_0 - x)^2 = 0$$

(110), (120), (210), (220), (330), (331), (131), (231)

Order 12. Triquaternions.

Order 16. Quadriquaternions.

It is not a matter of much difficulty to work out many other cases, but the attention of the writer has not been called to any other cases which have been developed.

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[1] SCHEFFERS Q_2.

[2] SCHEFFERS Q_3, Q_4.

PART III. APPLICATIONS.

XXIII. GEOMETRICAL.

449. The chief geometrical applications of linear associative algebras have been in Quaternions, Octonions, Triquaternions, and Alternate Numbers. These will be sketched here very briefly, as the treatises on these subjects are very complete and easily accessible. What is usually called vector analysis may be found under these heads. There are two other algebras which find geometrical application in a way which may be extended to all algebras. These will be noticed immediately.[1]

450. Equipollences. The algebra of ordinary complex numbers

	e_0	e_1
e_0	e_0	e_1
e_1	e_1	$-e_0$

$e_0 = 1$

has been applied to the plane. To each point (x, y) corresponds a number $z = x + ye_1$. The analytic functions of z (say $f(z)$ where $df.z = f'(z).dz$) represent all conformal transformations of the plane; that is, if z traces any figure C_1 in the plane, $f(z)$ traces a figure C_2 such that every point of C_1 has a corresponding point on C_2 and conversely, and every angle in C_1 has an equal angle in C_2 and conversely.[2]

451. Equitangentials. The algebra

	e_0	e_1
e_0	e_0	e_1
e_1	e_1	0

$e_0 = 1$

has also been applied to the plane. The analytic functions of z represent the equisegmental transformations of the plane, such that $f(z)$ converts a figure into a second figure which preserves all lengths.[3] To $z = x + e_1 y$ corresponds the line $\zeta \cos x + \eta \sin x - y = 0$.

452. Quaternions. Three applications of Quaternions have been made to Geometry. In the *first* the vector of a quaternion is identified with a vector in space. The quotient or product of two such vectors is a quaternion whose axis is at right angles to the given vectors. Every quaternion may be expressed as the quotient of two vectors.

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[1] See *Bibliography of Quaternions*. Also the works of HAMILTON, CLIFFORD, COMBEBIAC, GRASSMANN, GIBBS and their successors.

[2] BELLAVITIS 1–16; SCHEFFERS 10.

[3] SCHEFFERS 10.

The following formulae are easily found:

(1) If α is parallel to β . . . . $V.\alpha\beta = 0$

(2) If α is perpendicular to β . . . . $S.\alpha\beta = 0$

(3) The plane through the extremity of δ, and perpendicular to α is . . . . $S(\rho - \delta)\,\alpha = 0$

(4) The line through the extremity of α, parallel to β . . . . $V(\rho - \alpha)\,\beta = 0$

(5) Equation of collinearity of α, β, γ . . . . $V(\alpha - \beta)\,(\beta - \gamma) = 0$

(6) Equation of coplanarity of $\alpha, \beta, \gamma, \delta$ . . . . $S(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \delta) = 0$

(7) Equation of concyclicity of $\alpha, \beta, \gamma, \delta$ . . . . $V(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \delta)\,(\delta - \alpha) = 0$

(8) Equation of cosphericity of $\alpha, \beta, \gamma, \delta, \varepsilon$ . . . . . $S(\alpha - \beta)\,(\beta - \gamma)\,(\gamma - \delta)\,(\delta - \varepsilon)\,(\varepsilon - \alpha) = 0$

(9) The operator $q\,(\;)\,q^{-1}$ turns the operand ( ) through the angle which is twice the angle of q, about the axis of q. The operand may be any expression, and thus turns like a rigid body. These operators give the group of all rotations.[1]

(10) The central quadric may be written $S\rho\phi\rho = -1 = g\rho^2 + 2\,S\lambda\rho\,S\mu\rho$, where ϕ is a linear vector self-transverse function; λ and μ are the cyclic normals;

$$t\lambda = \sqrt{g_2 - g_1}\; i + \sqrt{g_3 - g_2}\, k \qquad 2t^{-1}\mu = \sqrt{g_2 - g_1}\; i - \sqrt{g_3 - g_2}\, k$$

i and k being in the direction of the greatest and the least axes, and the axes are given by $g_1 = \dfrac{1}{a^2}$, $g_2 = \dfrac{1}{b^2}$, $g_3 = \dfrac{1}{c^2}$. Conjugate diameters are given by $S\alpha\phi\beta = S\beta\phi\gamma = S\gamma\phi\alpha = 0$.

(11) For any curve, $\rho = \phi\,(t)$, any surface, $\rho = \phi\,(t, u)$ or $F(\rho) = 0$. $d\rho$ is parallel to the tangent of a curve, $V\dfrac{d^2\rho}{d\rho\,Td\rho}$ is the vector curvature, $Ud\rho\;S\dfrac{d^3\rho}{Vd\rho\,d^2\rho}$ is the vector torsion, $\alpha = Ud\rho$ is the unit tangent, $\beta = UVd\rho\,d^2\rho\,Ud\rho$ is a unit on the principal normal, $\gamma = UVd\rho\,d^2\rho$ is a unit on the binormal. For a surface $F(\rho) = 0$, ∇F is the normal, $S(\rho - \rho_0)\,\nabla F_0 = 0$ is the tangent plane.[2]

The *second* application[3] of quaternions to geometry is by a homogeneous method. In this the quaternion q is written $q = Sq\,(1 + \rho)$, and q is regarded as the affix of the point ρ with a weight $Sq = w$.

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[1] CAYLEY 10.

[2] HAMILTON'S works, TAIT'S works, JOLY'S works.

[3] This application may be followed in JOLY 20, 11, 25; SHAW 3; CHAPMAN 4; see also BRILL 1.

We write also

$$A\,.\,q\,Ars = \begin{vmatrix} r & s \\ S\bar{q}r & S\bar{q}s \end{vmatrix}$$

$$A\,.\,qr\,Astu = \begin{vmatrix} s & t & u \\ S\bar{q}s & S\bar{q}t & S\bar{q}u \\ S\bar{r}s & S\bar{r}t & S\bar{r}u \end{vmatrix}$$

$$A\,.\,qrs\,A\,.\,tuvw = \begin{vmatrix} t & u & v & w \\ S\bar{q}t & S\bar{q}u & S\bar{q}v & S\bar{q}w \\ S\bar{r}t & S\bar{r}u & S\bar{r}v & S\bar{r}w \\ S\bar{s}t & S\bar{s}u & S\bar{s}v & S\bar{s}w \end{vmatrix}$$

In particular we may write

$$-A\,.\,1\,Aab = A'\,.\,ab$$
$$A\,.\,ab\,Aijk = A''.\,ab = V\,.\,Va\,Vb$$
$$S\,.\,\bar{a}\,Abc\,Aijk = S\,.\,Va\,Vb\,Vc = SA\,.\,abc$$
$$A\,.\,abc = -K\,.\,A\,.\,abc\,A1\,ijk$$
$$S\,.\,\bar{a}\,A\,.\,bcd = -S\bar{a}\,Abcd\,A\,.\,1\,ijk$$

We have

(1) The equation of line a, b is $A\,.\,abq = 0$.

(2) The equation of plane a, b, c is $S\,.\,\bar{q}\,Aabc = 0$.

(3) a, b and c, d intersect if $S\,.\,\bar{a}\,Abcd = 0$.

(4) The point of intersection of $S\,.\,lq = 0 = Smq = Snq$ is $q = A\,.\,lmn$.

The *third* application of quaternions is to four-dimensional space.[1]

(1) Any quaternion p represents a four-dimensional vector in parabolic space. All vectors parallel to p, in the same sense, and equal in length are represented by p.

(2) If q is a second vector, then the angle $\angle\,(p, q)$ being θ

$$\cos\theta = SUp\,\overline{Uq} = S\,.\,\overline{Up}\,Uq$$

(3) The condition that p is perpendicular to q is $S\bar{p}q = Sp\bar{q} = 0$.

(4) There is for p as a multiplier $p\,()$ a system of invariant planes, one through any given line q, called a system of *in-parallel planes*. Multiplication by p, $()\,p$, has also a system of invariant planes, called *by-parallel planes*, one through each line q. The displacement of q in any invariant plane is constant and equal to the angle of p. The tensor of q is multiplied by the tensor of p. If q is resolved parallel to two invariant planes of p, these components turn in their planes through $\angle\,p$, and the product pq has these results for its components.

(5) If $Vq\bar{p} = 0$, q is parallel to p.

(6) The projection of q on p is $Up\,Sq\,KUp$.

The projection of q on a vector perpendicular to p is $Vq\,KUp\,.\,Up$.

(7) The plane through the origin and the two vectors from the origin

$$\alpha_1 - \alpha_2 \text{ and } \alpha_1(\alpha_1 - \alpha_2) \text{ is } \alpha_1 p + p\alpha_2 = 0$$

[1] Hathaway 2, 3, 4, 5; Stringham 4, 5, 7.

The plane through the point $\alpha_1\, a$ containing the vectors

(8) If
$$\alpha_1 - \alpha_2 \text{ and } \alpha_1(\alpha_1 - \alpha_2) \text{ is } \alpha_1 q + q\alpha_2 + 2a = 0$$
$$\alpha_1 = \pm\, UVe\bar{c}, \quad \alpha_2 = \pm\, UV\bar{e}c, \text{ and } a = -\alpha_1 a_0$$

then the equation of the plane through a_0 containing the vectors c, e is
$$\alpha_1 p + p\alpha_2 + 2a = 0$$

(9) The plane through c, d, e is given by the same equation with
$$\alpha_1 = UV(c\bar{d} + d\bar{e} + e\bar{c}) \qquad \alpha_2 = UV(\bar{c}d + \bar{d}e + \bar{e}c)$$
$$a = -\tfrac{1}{2}(\alpha_1 c + c\alpha_2) = -\tfrac{1}{2}(\alpha_1 d + d\alpha_2) = -\tfrac{1}{2}(\alpha_1 e + e\alpha_2)$$

(10) The normal to the plane is $\alpha_1\, a$.

(11) The point of intersection of the two planes
$$\alpha_1 p + p\alpha_2 + 2a = 0 = \beta_1 p + p\beta_2 + 2b$$
is
$$p = \frac{\beta_1 a - a\beta_2 + \alpha_1 b - b\alpha_2}{S(\alpha_2\beta_2 - \alpha_1\beta_1)}$$

(12) If the two planes through the origin $(\alpha_1\alpha_2 0)$ $(\beta_1\beta_2 0)$ meet in a line through the origin, it is necessary and sufficient that
$$S\alpha_1\beta_1 = S\alpha_2\beta_2$$
The cosine of the dihedral angle between the planes is $\pm S\alpha_1\beta_1 = \pm S\alpha_2\beta_2$. They are perpendicular when this vanishes.

(13) The two planes $(\alpha_1\alpha_2 2a)$ $(\beta_1\beta_2 2b)$ meet in a straight line if
$$\alpha_1 b - b\,\alpha_2 + \beta_1 a - a\,\beta_2 = 0$$
Let
$$f = \alpha_1 b - b\alpha_2 \qquad g = \beta_1 a - a\beta_2 \qquad m = S(\alpha_2\beta_2 - \alpha_1\beta_1)$$
then if $f = -g \mp 0$, the equation of this line is
$$p = \frac{x - 2Vab}{f}$$

(14) The two planes meet in a point at infinity if $m = 0$ and $f + g \mp 0$; they meet in a line at infinity if
$$\beta_1 = \pm\,\alpha_1 \qquad \beta_2 = \pm\,\alpha_2$$

(15) The perpendicular distance between the planes $(\alpha_1\alpha_2 2a)$ $(\alpha_1\alpha_2 2b)$ is in magnitude and direction $\alpha_1(a - b)$.

(16) The vector normal from the extremity of c to the plane $(\alpha_1\alpha_2 2b)$ is
$$\tfrac{1}{2}\alpha_1(2a + \alpha_1 c + c\alpha_2)$$

(17) The vector normal from the origin to the intersection of
$$(\alpha_1\alpha_2 2a) \text{ and } (\beta_1\beta_2 2b) \text{ is } \frac{a\bar{b} - b\bar{a}}{\bar{b}\alpha_1 - \alpha_2\bar{b}} \qquad f + g = 0,\ f \mp g$$

(18) Two planes meet in general in a point or in a straight line. Through any common point transversal planes may be passed meeting the two in two

straight lines u, v and forming with them equal opposite interior dihedral angles. The angle between these lines u, v is the *isoclinal angle* of the two planes. Two planes have maximal and minimal isoclinal angles if there exist solutions $c + \alpha_1^\theta u$ and $c + \beta_1^\theta v$ of their equations such that

$$S\alpha_1 u\bar{v} = 0 \qquad S\beta_1 u\bar{v} = 0 \qquad Su\bar{v} \mp S\alpha_1 u \overline{\beta_1 v}$$

The planes of these angles and these only cut the given planes orthogonally. The lines u and v are given by

$$u = \alpha_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\alpha_2 \qquad \gamma_1 = UV\alpha_1\beta$$
$$v = \beta_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\beta_2 \qquad \gamma_2 = UV\alpha_2\beta_1$$
$$\alpha_1 u = u' = \alpha_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)\alpha_2$$
$$\beta_1 v = v' = \beta_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)\beta_2$$

(19) There are no maximal and minimal isoclinal angles if any one of the four conditions is satisfied:

$$\beta_1 = \pm \alpha_1 \qquad \beta_2 = \pm \alpha_2$$

In this case the isoclinal angle is constant for all variations of θ.

(20) Two planes are perpendicular and meet in a point if

$$S\alpha_1\beta_1 = -S\alpha_2\beta_2 \mp 0 \text{ or } \mp 1$$

Two planes are perpendicular and meet in a line if

$$S\alpha_1\beta_1 = S\alpha_2\beta_2 = 0$$

Two planes are hyperpendicular if every line in one is perpendicular to every line in the other. In this case

$$S\alpha_1\beta_1 = -S\alpha_2\beta_2 = \pm 1$$

that is

$$\beta_1 = \pm \alpha_1 \qquad \beta_2 = \mp \alpha_2$$

If two planes are parallel

$$\alpha_1 = \beta_1 \qquad \alpha_2 = \beta_2$$

453. Octonions. The following are the simpler results:

(1) The vector from O to P is a rotor ρ and may be transferred anywhere along its own line. It is not equal to any parallel rotor. Rotors from the same point O are added like vectors, $\rho + \varepsilon$ being the diagonal of the parallelogram whose sides *from* O are ρ and ε.

(2) The side parallel to ρ is $\rho + \Omega M\varepsilon\rho$, that parallel to ε is $\varepsilon + \Omega M\rho\varepsilon$.

(3) If all vectors are drawn from O, the usual formulae of quaternions hold. Thus the equation of the plane perpendicular to δ through its extremity is $S(\rho - \delta)\delta = 0$; the line through the extremity of δ parallel to α is $\rho = \delta + t\alpha$. But a rotor *in* the plane is not $\rho - \delta$ but $\rho - \delta + \Omega M . \delta\rho$ and a rotor on the line is not $x\alpha$ but $x(\alpha + \Omega M\delta\alpha)$.

(4) A velocity of rotation about an axis is represented by a rotor on that axis, a translation along the axis is a lator on that axis. A motor, as $\omega + \Omega\sigma$, indicates a displacement such that in time dt any point rotates about the axis of the motor by an angle $T\omega . dt$ and is translated along the axis by a distance $T\sigma dt$.

(5) The axis of $M.AB$ is the common perpendicular of the axes of A and B. The rotor of $M.AB$ is the vector of the product of the rotors of A and B considered as vectors through O. The lator of $M.AB$ has a pitch equal to the sum of the pitches of A and B and the length of the common perpendicular multiplied by the cotangent of the angle between A and B ($= d \cot \theta$).

(6) The rotor of $A + B$ is equal and parallel to $\alpha_1 + \beta$, the sum of two rotors from O equal to the rotor of A and parallel to the rotor of B respectively, and intersects the common perpendicular from A to B at a distance from O equal to [ϖ being the common perpendicular]

$$T\left[\varpi S\beta(\alpha_1 + \beta)^{-1} + (p - p')M\alpha_1\beta \,.\, (\alpha_1 + \beta)^{-2}\right]$$

(7) $S_1 . ABC$ is one-sixth of the volume of the parallelopiped whose edges are the rotors of ABC. $M_1 . ABC$ is a rotor determined from the rotors of A, B, C as $V.\alpha\beta\gamma$ is from α, β, γ.

$t.S.ABC = tA + tB + tC + d\cot\theta - e\tan\phi$; d and θ as in (5), e and ϕ the common perpendicular from $M.AB$ to C, and the angle.

$$t.MABC = tA + tB + tC - \frac{d\cot\theta - e\tan\phi}{\cot^2\theta\tan^2\phi + \cot^2\theta + \tan^2\phi}$$

(8) If B and C are motors whose rotors are not zero and not parallel, then $XB + YC$ is any motor which intersects the common perpendicular of B and C perpendicularly.[1]

454. **Triquaternions.** If μ, μ' are points, δ, δ' lines, ϖ and ϖ' planes, all of unit tensor,

$\mu = \mu x_0 + \omega(ix_1 + jx_2 + kx_3)$ μ is the point $\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}$

$\varpi = \omega\beta_0 + \mu(i\alpha_1 + j\alpha_2 + k\alpha_3)$ ϖ is the plane $\beta_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

$\delta = i\alpha_1 + j\alpha_2 + k\alpha_3 + \omega(i\beta_1 + j\beta_2 + k\beta_3)$ $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$

δ is the line $\frac{p_{01}}{\alpha_1} : \frac{p_{02}}{\alpha_2} : \frac{p_{03}}{\alpha_3} : \frac{p_{23}}{\beta_1} : \frac{p_{31}}{\beta_2} : \frac{p_{12}}{\beta_3}$

That is, a point or a plane is represented by the symmetry transformation it produces; a line, by a rotation about it as an axis through 180°.

(1) $G\delta\delta'$ is — cos of angle between the lines.

(2) $G\varpi\varpi'$ is — cos of angle between the planes.

(3) $L\mu\mu'$ is the vector of μ' towards μ.

(4) $L\mu\delta$ is the vector perpendicular of the plane containing the point and the line, tensor equal to distance from point to line.

(5) $L\delta\delta'$ is the complex whose axis is the common perpendicular and whose automoment is the product of the shortest distance by the cotangent of the angle.

(6) $L\mu\varpi$ is the perpendicular drawn through μ to the plane ϖ.

(7) $L\delta\varpi$ is the point of intersection of the line and the plane, tensor equal to the sine of the angle of the line and plane.

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[1] M'Aulay 2.

(8) $L\varpi\varpi'$ is the line of intersection of the two planes, tensor equal to the sine of the angle.

(9) $P\mu\delta$ is the plane through μ perpendicular to δ.

(10) $P\delta\delta'$ is the plane at ∞ multiplied by the shortest distance and the sine of the angle of the two lines. $\mathrm{P}\delta\delta'$ is the moment of the two lines.

(11) $P\mu\varpi$ is the plane at ϖ multiplied by the distance from μ to ϖ and positive or negative as μ is on the side of the positive or negative aspect of the plane.

(12) $P\delta\varpi$ is the plane drawn through the line perpendicular to the plane, tensor equal to the sine of the angle of the line and the positive normal of the plane.

(13) If γ, γ' are two complexes of unit tensors, $\mathrm{P}\gamma\gamma' = 0$ means the two are in involution.

(14) A displacement without deformation is given by $r(\,)r^{-1}$:

$$r = q + \omega q_1 \qquad Sq\bar{q}_1 = 0 \qquad \mathrm{P}\,[r^2 - (Lr)^2] = 0$$

The axis is $\delta = VLr = U(Vq + \omega Vq_1)$.

The angle of rotation is 2θ. $\quad \theta = \tan^{-1}\dfrac{Sq}{TVq}$

The translation is 2η. $\quad \eta = -\dfrac{Sq_1}{TVq}$

$$r = (1 + \omega\delta\eta)\,(\cos\theta + \delta\sin\theta)$$

(15) Transformations by similitude are given by $r = \mu q + \omega q_1$. $\quad Sq\bar{q}_1 = 0$

(16) The triquaternion r produces a point transformation $m' = rmr^{-1}$,

if $r = w + l + p$, $\qquad 2wp - \mathrm{P}\,l^2 = 0$

This transformation may be written $\dfrac{(w + m)\;(w + d)}{w}$, which is a rotation about the line d, and a homothetic transformation whose center is m and coefficient $\dfrac{w - Tm}{w + Tm}$.

Hence r produces the group of transformations by similitude.[1]

(17) A sphere[2] is represented by the inversion which it leaves invariant; that is, by the quadriquaternion $\mu\,(ix_1 + jy_1 + kz_1) + \omega w_2 + \omega' w_3$.

(18) If M and M' are two spheres of zero radius, m and m' their centers, $L\,m\,M' = L\,m'\,M$ is the line $(m\,m')$. The sphere on $m\,m'$ as diameter is $P\,m\,M'$. If d is a line, then $\mathrm{P}\,.\,Md$ is the plane through d and m.

455. Alternates. There are various applications of the different systems of alternates, notably those which are called *space-analysis*—the development of Grassmann's systems; *vector-analysis*—a Grassmann system without the use of point-symbols or else a system due to Gibbs; and finally the Clifford systems. No brief account or exhibition of formulæ can be given.[3]

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[1] Combebiac 2. [2] Combebiac 3.

[3] See *Bibliography of Quaternions*; notably Joly 6; Hyde 4; Whitehead 1; Gibbs–Wilson 3.

XXIV. PHYSICO-MECHANICAL APPLICATIONS.

456. These are so numerous that they may be only glanced at. Quaternions has been applied to all branches of mechanics and physics, biquaternions and triquaternions to certain parts of mechanics and physics, alternates and vector analysis in general to mechanics and physics. The standard treatises already mentioned may be consulted.

XXV. TRANSFORMATION GROUPS.[1]

457. Theorem. To every linear associative algebra containing a modulus belongs a simply transitive group of linear homogeneous transformations, in whose finite equations the parameters appear linearly and homogeneously, and conversely.[2]

458. Theorem. Associated with every linear associative algebra containing a modulus and of order r, is a pair of reciprocal simply transitive linear homogeneous groups in r variables.[3]

459. Theorem. To a simply transitive bilinear group which has the equations

$$X_i f = \sum_{k,s} \alpha_{kis} x_k \frac{\partial f}{\partial x_s} \qquad (i = 1 \ldots . r)$$

$$X_i X_k . f = \sum_s \alpha_{iks} X_s f$$

corresponds the algebra whose multiplication table is $e_i e_k = \sum_s \alpha_{iks} e_s$, and conversely.[4]

460. Theorem. The product of $a = \sum_{i=1}^{r} a_i e_i$ and $b = \sum_{i=1}^{r} b_i e_i$ gives the finite transformation corresponding to the successive transformations[5] of the parameters $(a_1 \ldots . a_r)$ and $(b_1 \ldots . b_r)$.

461. Theorem. To every sub-group of G, the group corresponding to the algebra Σ, corresponds a sub-algebra of Σ, and conversely. To every invariant sub-algebra of Σ corresponds an invariant sub-group[4] of G.

462. Theorem. To the nilpotent sub-algebra of Σ corresponds a sub-group of G, $Y_1(f) \ldots . Y_k(f)$, such that for no values of $Y.f$ or $X.f$, transformations respectively of the sub-group and the group, do we have[6]

$$Y(Xf) = \omega Xf \qquad \omega \neq 0$$

$$X(Yf) = \omega' Xf \qquad \omega' \neq 0$$

$$(YX) = Y(Xf) - X(Yf) = \omega Xf \qquad \omega \neq 0$$

463. Theorem. The invariant sub-group g, corresponding to the nilpotent sub-algebra σ, is of rank zero.[4]

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[1] STUDY 7. [2] POINCARÉ 1, 2, 3; STUDY 3; CARTAN 2. See also SCHUR 1.
[3] STUDY 1, 3; LIE–SCHEFFERS 4; CARTAN 2. [4] CARTAN 2. [5] CARTAN 2; STUDY 3.
[6] CARTAN 2. Cf. ENGEL, Kleinere Beiträge zur Gruppentheorie, *Leipziger Berichte*, 1887, S. 96; 1893, S. 360–369.

464. Theorem. To every quadrate of order $r = p^2$ corresponds the parameter group of the linear homogeneous group of p variables.[1]

465. Theorem. To every SCHEFFERS or PEIRCE algebra corresponds an integrable simply transitive bilinear group, whose infinitesimal transformations are

$$X_i = x_i \frac{\partial}{\partial x_i} + \sum_\rho y_\rho \frac{\partial}{\partial y_\rho} \qquad i = \beta_\rho \quad (\alpha_i, \beta_i \text{ are the } \textit{characters} \text{ of } \gamma_i)$$

$$Y_i = x_{\alpha_i} \frac{\partial}{\partial x_i} + \sum_{j,s} \alpha_{jis}\, y_j \frac{\partial}{\partial y_s} \qquad (s > i,\ s > j)$$

and whose finite equations are[2]

$$x_i' = a_i\, x_i$$
$$y_i' = a_{\beta_i} y_i + b_i\, x_{\alpha_i} + \sum_{\lambda,\mu} \alpha_{\lambda\mu i} b_\mu\, y_\lambda \qquad (\lambda < i,\ \mu < i)$$

466. Theorem. Every simply transitive group can be deduced from a group of the form just given,

$$\begin{cases} X^{(i)} = X^{(i)} \dfrac{\partial}{\partial X^{(i)}} + \sum\limits_\rho Y^{(\rho)} \dfrac{\partial}{\partial Y^{(\rho)}} & (i = \beta_\rho) \\ Y^{(i)} = X^{(\alpha_i)} \dfrac{\partial}{\partial Y^{(i)}} + \sum\limits_{j,s} \alpha_{jis}\, Y^{(j)} \dfrac{\partial}{\partial Y^{(s)}} & (s > i,\ s > j,\ \beta_j = \alpha_i,\ \alpha_s = \alpha_j,\ \beta_s = \beta_i) \end{cases}$$

or

$$\begin{cases} X'^{(i)} = A^{(i)}\, X^{(i)} \\ Y'^{(i)} = A^{(\beta_i)}\, Y^{(i)} + B^{(i)}\, X^{(\alpha_i)} + \sum\limits_{\lambda,\mu} \alpha_{\lambda\mu i} B^{(\mu)}\, Y^{(\lambda)} & (\lambda < i,\ \mu < i) \end{cases}$$

by setting to correspond to each variable $X^{(\rho)}$ or $Y^{(\rho)}$ of character $(\alpha\beta)$, $p_\alpha\, p_\beta$ new variables $x_{ij}^{(\rho)}$, $y_{ij}^{(\rho)}$, where i, j are respectively any two numbers of the series $1, 2 \ldots\ldots p_\alpha$, $1, 2 \ldots\ldots p_\beta$. Likewise to each parameter $A^{(\rho)}$, $B^{(\rho)}$ of character $(\alpha\beta)$, $p_\alpha\, p_\beta$ new parameters $a_{ij}^{(\rho)}$, $b_{ij}^{(\rho)}$.

The simply transitive group is then defined by the infinitesimal transformations

$$X_{\alpha\beta}^{(i)} = \sum_{\lambda=1}^{p_i} x_{\lambda\alpha}^{(i)} \frac{\partial}{\partial x_{\lambda\beta}^{(i)}} + \sum_{\rho,\lambda}^{\lambda=1\ldots\ldots p_\alpha} y_{\lambda\alpha}^{(\rho)} \frac{\partial}{\partial y_{\lambda\beta}^{(\rho)}} \qquad (\beta_\rho = i;\ \alpha, \beta = 1, 2 \ldots\ldots p_i)$$

$$Y_{\alpha\beta}^{(i)} = \sum_{\lambda=1}^{p_i} x_{\lambda\alpha}^{(\alpha_i)} \frac{\partial}{\partial y_{\lambda\beta}^{(i)}} + \sum_{j,s,\lambda} \alpha_{jis}\, y_{\lambda\alpha}^{(j)} \frac{\partial}{\partial y_{\lambda\beta}^{(s)}}$$

or by the finite equations[2]

$$x'^{(i)}_{\alpha\beta} = \sum_\lambda^{1,2\ldots\ldots p_i} a_{\lambda\beta}^{(i)}\, x_{\alpha\lambda}^{(i)}$$

$$y'^{(i)}_{\alpha\beta} = \sum_\lambda a_{\lambda\beta}^{(\beta_i)}\, y_{\alpha\lambda}^{(i)} + \sum_\lambda b_{\lambda\beta}^{(i)}\, x_{\alpha\lambda}^{(\alpha_i)} + \sum_{\rho\sigma\lambda} \alpha_{\rho\sigma i}\, b_{\lambda\beta}^{(\sigma)}\, y_{\alpha\lambda}^{(\rho)}$$

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[1] CARTAN 2; MOLIEN 1. Cf. CAYLEY 11, 5; LAGUERRE 1; STEPHANOS 1; KLEIN 1; LIPSCHITZ 2. Also CAYLEY 3; FROBENIUS 1; SYLVESTER 1; WEYR 5, 6, 7, 8.

[2] CARTAN 2.

467. Theorem. Every simply transitive bilinear group G is formed of a sub-group Γ of rank zero, and a sub-group g which is composed of h groups $g_1 \dots g_h$, respectively isomorphic with general linear homogeneous groups of $p_1, p_2 \dots p_h$ variables. Moreover the variables may be so chosen that the p_1^2 first variables are interchanged by the first g_1 of these h groups, like the parameters of the general linear homogeneous group on p_1 variables, and are not altered by the other $h-1$ groups; the same is true of the $p_2^2 \dots p_h^2$ following variables; finally these $p_1^2 + \dots + p_h^2$ variables are not changed by the sub-group[1] Γ.

468. Theorem. All simply transitive groups are known when those in § 465 are known.[2]

469. Theorem. Every real simply transitive group G is composed of an invariant sub-group Γ of rank zero, and a sub-group g which is the sum of h groups $g_1 \dots g_h$, each of which belongs to one or other of the three types following:

(1) The groups of the first type are on p^2 variables x_{ij} and are given by the formulæ

$$X_{ij} = x_{1i}\frac{\partial}{\partial x_{1j}} + x_{2i}\frac{\partial}{\partial x_{2j}} + \dots + x_{pi}\frac{\partial}{\partial x_{pj}}$$

or

$$x'_{ij} = a_{ij}\,x_{i1} + a_{2j}\,x_{i2} + \dots + a_{pj}\,x_{ip}$$

giving the parameter group of the general linear homogeneous group on p variables.

(2) The groups of the second type are on the $2p^2$ variables x_{ij}, y_{ij} and are given by the formulæ

$$X_{ij} = x_{1i}\frac{\partial}{\partial x_{1j}} + \dots + x_{pi}\frac{\partial}{\partial x_{pj}} + y_{1i}\frac{\partial}{\partial y_{1j}} + \dots + y_{pi}\frac{\partial}{\partial y_{pj}}$$

$$Y_{ij} = x_{1i}\frac{\partial}{\partial y_{1j}} + \dots + x_{pi}\frac{\partial}{\partial y_{pj}} - y_{1i}\frac{\partial}{\partial x_{1j}} + \dots - y_{pi}\frac{\partial}{\partial x_{pj}}$$

or

$$x'_{ij} = a_{1j}\,x_{i1} + \dots + a_{pj}\,x_{ip} - b_{1j}\,y_{i1} - \dots - b_{pj}\,y_{ip}$$
$$y'_{ij} = a_{1j}\,y_{i1} + \dots + a_{pj}\,y_{ip} + b_{1j}\,x_{i1} + \dots + b_{pj}\,x_{ip}$$

(3) Those of the third type are on $4p^2$ variables x_{ij}, y_{ij}, z_{ij}, t_{ij}, given by the formulæ

$$X_{ij} = \sum_{\lambda=1}^{p}\left(x_{\lambda i}\frac{\partial}{\partial x_{\lambda j}} + y_{\lambda i}\frac{\partial}{\partial y_{\lambda j}} + z_{\lambda i}\frac{\partial}{\partial z_{\lambda j}} + t_{\lambda i}\frac{\partial}{\partial t_{\lambda j}}\right)$$

$$Y_{ij} = \sum_{\lambda=1}^{p}\left(x_{\lambda i}\frac{\partial}{\partial y_{\lambda j}} - y_{\lambda i}\frac{\partial}{\partial x_{\lambda j}} - z_{\lambda i}\frac{\partial}{\partial t_{\lambda j}} + t_{\lambda i}\frac{\partial}{\partial z_{\lambda j}}\right)$$

$$Z_{ij} = \sum_{\lambda=1}^{p}\left(x_{\lambda i}\frac{\partial}{\partial z_{\lambda j}} + y_{\lambda i}\frac{\partial}{\partial t_{\lambda j}} - z_{\lambda i}\frac{\partial}{\partial x_{\lambda j}} - t_{\lambda i}\frac{\partial}{\partial y_{\lambda j}}\right)$$

$$T_{ij} = \sum_{\lambda=1}^{p}\left(x_{\lambda i}\frac{\partial}{\partial t_{\lambda j}} - y_{\lambda i}\frac{\partial}{\partial z_{\lambda j}} + z_{\lambda i}\frac{\partial}{\partial y_{\lambda j}} - t_{\lambda i}\frac{\partial}{\partial x_{\lambda j}}\right)$$

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[1] CARTAN 2. Cf. MOLIEN 1.

[2] CARTAN 2.

or

$$x'_{ij} = \sum_{\lambda=1}^{p} (a_{\lambda j}\, x_{i\lambda} - b_{\lambda j}\, y_{i\lambda} - c_{\lambda j}\, z_{i\lambda} - d_{\lambda j}\, t_{i\lambda})$$

$$y'_{ij} = \sum_{\lambda=1}^{p} (a_{\lambda j}\, y_{i\lambda} + b_{\lambda j}\, x_{i\lambda} - c_{\lambda j}\, t_{i\lambda} - d_{\lambda j}\, z_{i\lambda})$$

$$z'_{ij} = \sum_{\lambda=1}^{p} (a_{\lambda j}\, z_{i\lambda} + b_{\lambda j}\, t_{i\lambda} + c_{\lambda j}\, x_{i\lambda} - d_{\lambda j}\, y_{i\lambda})$$

$$t'_{ij} = \sum_{\lambda=1}^{p} (a_{\lambda j}\, t_{i\lambda} - b_{\lambda j}\, z_{i\lambda} + c_{\lambda j}\, y_{i\lambda} + d_{\lambda j}\, x_{i\lambda})$$

To each of these groups of p^2, $2p^2$, or $4p^2$ variables we can set to correspond p^2, $2p^2$, or $4p^2$ variables which are interchanged by these equations, without being changed by the other groups which enter g nor by the sub-group Γ. All these variables are independent.[1]

470. Theorem. The groups in § 469 are not simple, but are composed of an invariant sub-group of one parameter and simple invariant sub-groups of $p^2 - 1$, $2p^2 - 1$, $4p^2 - 1$ parameters.[1]

471. Theorem. Simply transitive bilinear groups in involution (transformations commutative) are given by the formulæ

(1) $$X = x\frac{\partial}{\partial x} + \sum_i y_i \frac{\partial}{\partial y_i} \qquad Y_i = x\frac{\partial}{\partial y_i} + \sum_{\lambda, s} \alpha_{\lambda i s}\, y_\lambda \frac{\partial}{\partial y_s}$$

$$i = 1, 2 \ldots . r-1 \qquad \alpha_{\lambda i s} = 0 \ \text{ if } \ s \leqq i,\ s \leqq \lambda$$

or

$$x' = ax \qquad y'_i = ay_i + b_i x + \sum_{\lambda, \mu} \alpha_{\lambda\mu i}\, b_\mu\, y_\lambda$$

(2) $$X = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial x} + \sum_i \left(y_i \frac{\partial}{\partial y_i} + t_i \frac{\partial}{\partial t_i}\right)$$

$$Z = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial x} + \sum_i \left(y_i \frac{\partial}{\partial t_i} + t_i \frac{\partial}{\partial y_i}\right)$$

$$Y_i = x\frac{\partial}{\partial y_i} + z\frac{\partial}{\partial t_i} + \sum_{\lambda, s} (\alpha_{\lambda i s}\, y_\lambda - \beta_{\lambda i s}\, t_\lambda) \frac{\partial}{\partial y_s} + \sum_{\lambda, s} (\beta_{\lambda i s}\, y_\lambda + \alpha_{\lambda i s}\, t_\lambda) \frac{\partial}{\partial t_s}$$

$$T_i = x\frac{\partial}{\partial t_i} + z\frac{\partial}{\partial y_i} + \sum_{\lambda, s} (\beta_{\lambda i s}\, y_\lambda + \alpha_{\lambda i s}\, t_\lambda) \frac{\partial}{\partial y_s} + \sum_{\lambda, s} (\alpha_{\lambda i s}\, y_\lambda - \beta_{\lambda i s}\, t_\lambda) \frac{\partial}{\partial t_s}$$

or[1]

$$x' = ax - cz$$
$$z' = az + cx$$
$$y'_i = ay_i - ct_i + b_i x - d_i z + \Sigma\, \alpha_{\lambda\mu i}\, (b_\mu\, y_\lambda - d_\mu\, t_\lambda) - \Sigma\, \beta_{\lambda\mu i} (b_\mu\, t_\lambda + d_\mu\, y_\lambda)$$
$$t'_i = at_i + cy_i + b_i z + d_i x + \Sigma\, \alpha_{\lambda\mu i}\, (b_\mu\, t_\lambda + d_\mu\, y_\lambda) + \Sigma\, \beta_{\lambda\mu i} (b_\mu\, y_\lambda - d_\mu\, t_\lambda)$$

472. Theorem. Every bilinear group G is composed of an invariant sub-group Γ of rank zero, and of a sub-group g which is the sum of a certain number h of groups which are respectively isomorphic with the general linear

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[1] CARTAN 2.

homogeneous group on $p_1, p_2 \ldots . p_h$ variables. Every real bilinear group G is composed of a real invariant sub-group Γ, and a sub-group g which is the sum of h groups each isomorphic with one of the three following groups:

(1) The general linear homogeneous group on p variables.

(2) The group on $2p^2$ parameters and $2p$ variables x_i, y_i:

$$x'_i = a_{1i}x_1 + \ldots . + a_{pi}x_p - b_{1i}y_1 - \ldots . - b_{pi}y_p$$
$$y'_i = a_{1i}y_1 + \ldots . + a_{pi}y_p + b_{1i}x_1 + \ldots . + b_{pi}x_p$$

(3) The group on $4p^2$ parameters and $4p$ variables x_i, y_i, z_i, t_i:

$$x'_i = \sum_{\lambda=1}^{p} (a_{\lambda i}x_\lambda - b_{\lambda i}y_\lambda - c_{\lambda i}z_\lambda - d_{\lambda i}t_\lambda)$$
$$y'_i = \sum_{\lambda=1}^{p} (a_{\lambda i}y_\lambda + b_{\lambda i}x_\lambda + c_{\lambda i}t_\lambda + d_{\lambda i}z_\lambda)$$
$$z'_i = \sum_{\lambda=1}^{p} (a_{\lambda i}z_\lambda + b_{\lambda i}t_\lambda + c_{\lambda i}x_\lambda + d_{\lambda i}y_\lambda)$$
$$t'_i = \sum_{\lambda=1}^{p} (a_{\lambda i}t_\lambda + b_{\lambda i}z_\lambda + c_{\lambda i}y_\lambda + d_{\lambda i}x_\lambda)$$

Each of these groups is formed of a simple invariant sub-group on $p^2 - 1$, $2p^2 - 1$, or $4p^2 - 1$ parameters and an invariant sub-group on one parameter.[1]

473. Theorem. Every bilinear group G is composed of an invariant sub-group Γ of rank zero, and one or more groups $g_1, g_2 \ldots .$ of which each g is, symbolically, the general linear homogeneous group of a certain number of variables $X_1 \ldots . X_p$, these variables being real, imaginary, or quaternions, and the p^2 parameters having the same nature,

$$X'^{(i)}_a = \sum_{\lambda=1}^{p} X^{(i)}_\lambda A_{\lambda a}$$

If the variables and the parameters of the bilinear group G are any imaginary quantities whatever, the group is composed of an invariant sub-group Γ, of rank zero, and of one or more sub-groups $g_1, g_2 \ldots .$ of which each g is the general linear homogeneous group of a certain number of series of p variables, of course imaginary.[1]

474. Theorem. The quaternion algebra is isomorphic with the group of rotations about a fixed point,[2] with the group of projective transformations on a line, and with the group of special linear transformations around a point in a plane.

475. Theorem. Biquaternions is isomorphic with the group of displacements in space without deformation.[3]

476. Theorem. Triquaternions is isomorphic with the group of displacements and transformations by similitude.[4] Quadriquaternions is isomorphic with the group of conformal transformations of space.

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[1] CARTAN 2. [2] CAYLEY 10; LAGUERRE 1; STEPHANOS 1; STRINGHAM 3; BEEZ 1.
[3] M'AULAY 2; COMBEBIAC 1; STUDY 5. [4] COMBEBIAC 2, 3.

XXVI. ABSTRACT GROUPS.

477. Theorem. Every abstract group is isomorphic with a FROBENIUS algebra of the same order as the group.[1]

478. Theorem. The expressions for the numbers of the FROBENIUS algebra corresponding to the group are determined by finding the sub-algebra consisting of all numbers commutative with every number of the algebra, then determining by linear expressions the partial moduli of the separate quadrates of the algebra, and then multiplying on the right and on the left by these partial moduli. Every number is thus separated into the parts that belong to the different quadrates. The parts for any quadrate of order r_i determine the r_i^2 quadrate units of the sub-algebra consisting of the quadrate, which determination is not unique. In terms of these $r = \sum_{i=1}^{p} r_i^2$ units all numbers of the algebra may be expressed.[2]

479. Theorem. The characteristic equation of a FROBENIUS algebra consisting of p quadrates is the product of p irreducible determinant factors. The pre-latent equation and the post-latent equation are identical and consist of the products of these p irreducible factors each to a power r_i equal to its order.[3]

480. Theorem. The linear factors of a FROBENIUS algebra correspond to numbers which are commutative with all numbers. The number of linear factors is the order of the quotient-group; that is, the order r divided by the order of the commutator sub-group.

481. Theorem. The single unit in each of the quadrates of order unity, may be found as one of the solutions, σ, of the equations

$$\zeta\sigma = \sigma\zeta = t\sigma \quad \text{for all } \zeta\text{'s}$$

For the ζ's it is sufficient to take the r numbers corresponding to the operators of the group. Thus if $\sigma = \Sigma \,.\, x_i e_i$, and if $e_i e_j = e_{ij}$, hence $e_i e_{i^{-1}j} = e_j$, we must have

$$t\sigma = \sigma \,.\, e_j = \Sigma \,.\, x_i e_{ij} = \Sigma\, x_{ij^{-1}} e_i \quad \text{for all } j\text{'s}$$

Hence

$$tx_i = x_{ij^{-1}}$$

If e_j is of order μ_j, $e_j^{\mu_j} = e_1 = 1$, then

$$x_{ij^{-1}} = tx_i \qquad x_{ij^{-2}} = t^2 x_i \,\ldots\, x_{ij^{-s}} = t^s x_i \qquad t^{\mu_j} = 1, \text{ or } x_i = 0$$

Hence

$$t = \cos\frac{2\pi n}{\mu_j} + \sqrt{-1}\,\sin\frac{2\pi n}{\mu_j} = t_j \qquad n = 1 \,\ldots\, \mu_j$$

since not all x_i vanish.

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[1] POINCARÉ 4; SHAW 6. This theorem follows at once from CARTAN 2. See also § 121.
[2] POINCARÉ 4; SHAW 6.
[3] FROBENIUS 14; SHAW 6.

Hence
$$\sigma = \Sigma\, x_i\, (e_i + t_j\, e_{ij}{-1} + t_j^2 e_{ij}{-2} + \ldots .)$$
$$= \Sigma\, x_i\, e_i\, (1 + t_j\, e_j{-1} + \ldots . + t_j^{\mu_j - 1}\, e_j) \qquad (j = 1 \ldots . r)$$

The subscripts i run through those values only which are given by the table $\{G\} = \{e_i\, e_j^s\}$. By operating on σ with other numbers e_k we establish other equalities among the x's and finally arrive at the units in question.

482. Theorem. The units in the quadrates of order 2^2 may be found as the solutions of the equations
$$\zeta^2\sigma = t_1\zeta\sigma + t_2\sigma \qquad \sigma\zeta^2 = t_1'\sigma\zeta + t_2'\sigma \qquad (\zeta \text{ any number})$$
We may state this also
$$(\zeta_1\,\zeta_2 + \zeta_2\,\zeta_1)\,\sigma - t_1\,\zeta_1\,\sigma - t_2\,\zeta_2\,\sigma + t_3\sigma = 0 \qquad (\zeta_1, \zeta_2 \text{ any numbers})$$
The units in the quadrates of higher orders may be found by similar equations.

483. Theorem. The numbers e_i, $i = 1 \ldots . r$, may be arranged in conjugate classes, the sum of all of those in any class being commutative with all numbers of the algebra. If these sums are $K_1, K_2 \ldots . K_h$, then
$$\sigma = \sum_{i=1}^{h} x_i\, K_i$$
The partial moduli of quadrates of order 1^2, are formed by operating on σ with all numbers and determining the coefficients to satisfy the equations
$$\zeta\sigma = t\sigma \qquad \sigma\zeta = t\sigma$$
The partial moduli of quadrates of order 2^2 and higher orders satisfy the equations of § 482.

484. Theorem. Every Abelian group of order n defines the FROBENIUS algebra[1]
$$e_i = \lambda_{ii0} \qquad (i = 1 \ldots . r)$$

485. Theorem. The dihedron groups, generated by e_1, e_2,
$$e_1^m = 1 = e_2^2 \qquad e_2\, e_1 = e_1^{m-1}\, e_2$$
define FROBENIUS algebras as follows:

When m is odd: Let $\omega^m = 1$, ω being a primitive root of unity, then the algebra is given by
$$\lambda_{110} \quad \lambda_{220} \quad \lambda_{2i-1,\,2i-1,\,0} \quad \lambda_{2i-1,\,2i,\,0} \quad \lambda_{2i,\,2i-1,\,0} \quad \lambda_{2i,\,2i,\,0} \qquad \left(i = 2 \ldots . \frac{m+1}{2}\right)$$
We notice that
$$e_1 = \lambda_{110} + \lambda_{220} + \Sigma\,(\omega^{-i}\,\lambda_{2i+1,\,2i+1,\,0} + \omega^i\,\lambda_{2i+2,\,2i+2,\,0}) \quad \left(i = 1 \ldots . \frac{m-1}{2}\right)$$
$$e_2 = \lambda_{110} - \lambda_{220} + \Sigma\,(\lambda_{2i+1,\,2i+2,\,0} + \lambda_{2i+2,\,2i+1,\,0}) \qquad \left(i = 1 \ldots . \frac{m-1}{2}\right)$$

[1] SHAW 6. This reference applies to the following sections.

When m is even, the algebra is given by

$$\lambda_{110} \quad \lambda_{220} \quad \lambda_{330} \quad \lambda_{440} \quad \lambda_{2i-1,\,2i-1,\,0} \quad \lambda_{2i-1,\,2i,\,0} \quad \lambda_{2i,\,2i-1,\,0} \quad \lambda_{2i,\,2i,\,0} \qquad \left(i=3\,\ldots\,\frac{m}{2}+1\right)$$

We notice that [1]

$$\left.\begin{aligned} e_1 &= \lambda_{110}+\lambda_{220}-\lambda_{330}-\lambda_{440}+\Sigma\,\omega^{-(i-2)}\,\lambda_{2i-1,\,2i-1,\,0} \\ &\qquad +\Sigma\,\omega^{(i-2)}\,\lambda_{2i,\,2i,\,0} \\ e_2 &= \lambda_{110}-\lambda_{220}+\lambda_{330}-\lambda_{440}+\Sigma\,(\lambda_{2i-1,\,2i,\,0}+\lambda_{2i,\,2i-1,\,0}) \end{aligned}\right\}\left(i=3\,\ldots\,\frac{m}{2}+1\right)$$

486. Theorem. The rotation groups, not dihedron groups, define the algebras given below:

(a) The tetrahedral group: generators e_1, e_2 $\qquad e_1^3=1=e_2^2=(e_1\,e_2)^3$
Let $\omega^3=1$. The FROBENIUS algebra is

$$\lambda_{110} \quad \lambda_{220} \quad \lambda_{330} \quad \lambda_{440} \quad \lambda_{450} \quad \lambda_{460} \quad \lambda_{540} \quad \lambda_{550} \quad \lambda_{560} \quad \lambda_{640} \quad \lambda_{650} \quad \lambda_{660}$$

$$e_1=\lambda_{110}+\omega^2\lambda_{220}+\omega\,\lambda_{330}+\lambda_{440}+\omega^2\lambda_{550}+\omega\,\lambda_{660}$$

$$\begin{aligned} e_2=\lambda_{110}+\lambda_{220}+\lambda_{330}-\tfrac{1}{3}\,(\lambda_{440}+\lambda_{550}+\lambda_{660})+\tfrac{2}{3}\,(\lambda_{450}+\lambda_{460}+\lambda_{540} \\ +\lambda_{560}+\lambda_{640}+\lambda_{650}) \end{aligned}$$

(b) The octahedral group: $e_1^4=1=e_2^3=(e_1\,e_2)^2$.
Let $\omega^4=1$. The algebra is

$$\begin{matrix} \lambda_{110} & \lambda_{220} & \lambda_{330} & \lambda_{440} & \lambda_{340} & \lambda_{430} & \lambda_{550} & \lambda_{660} & \lambda_{770} & \lambda_{560} & \lambda_{650} & \lambda_{570} \\ \lambda_{750} & \lambda_{670} & \lambda_{760} & \lambda_{880} & \lambda_{990} & \lambda_{\alpha\alpha0} & \lambda_{890} & \lambda_{980} & \lambda_{8\alpha0} & \lambda_{\alpha80} & \lambda_{9\alpha0} & \lambda_{\alpha90} \end{matrix}$$

$$e_1=\lambda_{110}-\lambda_{220}+\lambda_{330}-\lambda_{440}+\lambda_{550}+\omega^3\lambda_{660}+\omega\,\lambda_{770}-\omega\,\lambda_{880}+\omega^3\lambda_{990}+\omega\,\lambda_{\alpha\alpha0}$$

$$\begin{aligned} e_2=\lambda_{110}+\lambda_{220}-\tfrac{1}{2}\,\lambda_{330}+\tfrac{1}{2}\sqrt{-3}\,\lambda_{430}+\tfrac{1}{2}\,(1+\omega)\,\lambda_{560}+\tfrac{1}{2}\,(1-\omega)\,\lambda_{570} \\ +\tfrac{1}{2}\,(1-\omega)\,\lambda_{890}+\tfrac{1}{2}\,(1+\omega)\,\lambda_{8\alpha0}+\tfrac{1}{2}\sqrt{-3}\,\lambda_{340}-\tfrac{1}{2}\,\lambda_{440} \\ +\tfrac{1}{2}\,(1+\omega)\,\lambda_{650}-\tfrac{1}{2}\,\lambda_{660}+\tfrac{1}{2}\,\lambda_{670}+\tfrac{1}{2}\,(1-\omega)\,\lambda_{980}+\tfrac{1}{2}\,\omega\,\lambda_{990}+\tfrac{1}{2}\,\lambda_{9\alpha0} \\ +\tfrac{1}{2}\,(1-\omega)\,\lambda_{750}+\tfrac{1}{2}\,\lambda_{760}+\tfrac{1}{2}\,\omega\,\lambda_{770}+\tfrac{1}{2}\,(1+\omega)\,\lambda_{\alpha80}+\tfrac{1}{2}\,\lambda_{\alpha90}-\tfrac{1}{2}\,\omega\,\lambda_{\alpha\alpha0} \end{aligned}$$

(c) The icosahedral group: $e_1^5=1=e_2^3=(e_2\,e_1)^2$.
The algebra is $\lambda_{110}\ \ \lambda_{ij0}\ \ \lambda_{kl0}\ \ \lambda_{pq0}\ \ \lambda_{st0}$ where [2]

$$i,j=2,\,3,\,4 \qquad k,l=5,\,6,\,7 \qquad p,q=8,\,9,\,\alpha,\,\beta \qquad s,t=\gamma,\,\delta,\,\varepsilon,\,\zeta,\,\eta$$

487. Theorem. The group G_{168}, $e_1^7=1=e_2^3=(e_2\,e_1)^2$, defines the algebra

$$\lambda_{110} \quad \lambda_{ij0} \quad \lambda_{kl0} \quad \lambda_{pq0} \quad \lambda_{st0} \quad \lambda_{uv0}$$

where [3]

$$\begin{matrix} i,j=2,\,3,\,4 & k,l=5,\,6,\,7 & p,q=8,\,9,\,\alpha,\,\beta,\,\gamma,\,\delta \\ s,t=\varepsilon,\,\zeta,\,\eta,\,\theta,\,\iota,\,\varkappa,\,\lambda & & u,v=\mu,\,\nu,\,o,\,\pi,\,\rho,\,\sigma,\,\tau,\,\phi \end{matrix}$$

488. Theorem. The groups defined by the relations $e_1^a=1=e_2^c$, $e_2^{-1}\,e_1\,e_2=e_1^m$, m prime to a, give FROBENIUS algebras of order $r=ac$ which are sums of quadrates as follows:

$$a_\iota kg \quad \text{of order } 1 \qquad h_j\,k_j \quad \text{of order } g_j^2$$

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[1] SHAW 6. [2] FROBENIUS 3; DICKSON 4. [3] POINCARÉ 4.

where a_i is the highest common factor of $m - 1$ and a; g is the lowest exponent for which $m^g \equiv 1 \pmod{a}$; $c = kg$.

If a' is the smallest divisor of a, $\neq 1$, and $a = a_1 a'$, then $m^{g_1} \equiv 1 \pmod{a_1}$.

If a'' is the next smallest divisor of a, $a = a_2 a''$, $m^{g_2} \equiv 1 \pmod{a_2}$, and so on for all divisors of a; if also $\phi(N)$ is the totient of (N), then

$$\phi(a) = hg \quad \phi(a_1) = h_1 g_1 \ldots \phi(a_j) = h_j g_j \quad (j = 1, 2 \ldots i - 1, i + 1 \ldots p)$$

We notice that if ω is a primitive a-th root of unity, π a primitive c-th root of unity

$$e_1 = \Sigma\, \omega^{s_x^{(l)} m^j} \lambda_{jj0}^{(i, x, l)} \qquad e_2 = \Sigma\, \pi^i (\Sigma\, \lambda_{j, j+1, 0}^{(i, x, l)})$$

wherein

$$i = 1 \ldots k_x \qquad j = 1 \ldots g_x \qquad l = 1 \ldots h_x$$

The multiples of a_{p-x+1}, namely $v_t a_{p-x+1}$, where v_t is prime to a_x, are divided into h_x sets of g_x each; $s_x^{(l)}$ is the lowest in the l-th set, the set being $s_x^{(l)}, m s_x^{(l)} \ldots m^{g_x - 1} s_x^{(l)}$; and $j + 1$ is reduced modulo[1] g_x.

489. Theorem. The algebra defined by the groups $1 = e_3^n = e_2^n = e_1^n$ $e_2 e_1 = e_1 e_2 \quad e_3 e_1 = e_1 e_3 \quad e_2 e_3 = e_1 e_3 e_2$ is given by the forms λ occurring in the equations

$$e_1 = \Sigma\, \omega^{-l_x n_{p-x+1}} \lambda_{jj0}^{(x, i, k, l_x)}$$
$$e_2 = \Sigma\, \omega^{-i + j l_x n_{p-x+1}} \lambda_{jj0}^{(x, i, k, l_x)}$$
$$e_3 = \Sigma\, \omega^{k} \lambda_{j, j+1, 0}^{(x, i, k, l_x)}$$

where

$$x = p + 1, p \ldots 1, 0 \quad n_0 = n \quad n_{p+1} = 1 \quad k, i = 1 \ldots n_{p-x+1} \quad j = 1 \ldots n_x$$

l_x is any integer $< n_x$ and prime to n_x [has therefore $\phi(n_x)$ values], $j + 1$ is reduced modulo n_x; n_x is any divisor of n, the quotient[1] being n_{p-x+1}.

490. The papers of FROBENIUS and BURNSIDE on group-characteristics should be consulted.

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[1] SHAW 12.

XXVII. SPECIAL CLASSES OF GROUPS.

491. Since every group determines a FROBENIUS algebra, it is evident that this algebra might be used to determine the group and to serve in applications of the group. Since the group admits only of multiplication, the group properties become those of certain numbers in the algebra combined only by multiplication. However, if the group is a group of operators, or may be viewed as a group of operators, it may happen that the result of operating on a given operand may be additive, in which case the numbers of the algebra become operators. Examples are given below.

492. Substitutions. Since every abstract group of order r is isomorphic with one or more substitution groups on r letters or fewer, it follows that the permutations or substitutions of such groups may be expressed by numbers of the algebra corresponding to the abstract group. Thus a rational integral algebraic function P of n variables may be reduced to the form

$$P = \sum_{i=1}^{m} P_i$$

where P_i is expressible in the form

$$(A_1^{(i)} + A_2^{(i)} S_2 + A_3^{(i)} S_3 + \ldots\ldots + A_n^{(i)} S_n) F_i$$

where $A_j^{(i)}$ is a positive or negative numerical coefficient and S_j is a substitution of the symmetric group of the n variables. F_i is a rational integral algebraic function of the variables. All the substitutional properties of P_i are direct consequences of the form $(A_1^{(i)} + \ldots\ldots + A_n^{(i)} S_n)$. For example

$$P = \tfrac{1}{2} a_2 - \tfrac{1}{2} a_3 + 3a_1^2 a_2 - 3a_1^2 a_3 - \tfrac{1}{5} a_2^2 a_3 + \tfrac{1}{5} a_2 a_3^2 = P_1 + P_2$$

where

$$P_1 = \tfrac{1}{2} [1 - (a_2 a_3) + (a_1 a_3) - (a_1 a_2 a_3)] \cdot a_2$$
$$P_2 = [3 - \tfrac{1}{5} (a_1 a_2 a_3) - 3 (a_2 a_3) + \tfrac{1}{5} (a_1 a_3)] \cdot a_1^2 a_2$$

wherein the bracket expresses an operation. We may find solutions for equations such as

$$(1 + \sigma + \sigma^2 + \sigma^3) P = 0 \qquad \sigma = (abcd)$$

or other forms in which the parenthesis is any rational integral expression in terms of substitutions.

The solution of this particular case is $P = (1 - \sigma) F$, where F is any rational integral function. These equations are useful in the study of invariants.[1]

493. Linear Groups. A group of linear substitutions has corresponding to it an abstract group, such that if the generating substitutions of the linear group, H, are $\Sigma_1, \Sigma_2 \ldots\ldots \Sigma_p$, with certain relations $\Sigma_{i_1} \Sigma_{j_1} \cdots\cdots \Sigma_{r_1} = 1$, $\Sigma_{i_2} \Sigma_{j_2} \ldots\ldots \Sigma_{r_2} = 1$, etc., then the abstract group is determined by generating substitutions $\sigma_1, \sigma_2 \ldots\ldots \sigma_p$, with relations $\sigma_{i_1} \sigma_{j_1} \cdots\cdots \sigma_{r_1} = 1$, $\sigma_{i_2} \sigma_{j_2} \cdots\cdots \sigma_{r_2} = 1$, etc.

[1] A. YOUNG 1.

If we choose a suitable polygon in a fundamental circle, the circle is divisible into an infinity of triangles, which may be produced by inversions at the corners of the polygon, according to the well-known methods. The group G generated by $\sigma_1, \sigma_2 \ldots \sigma_p$ without the relations is in general infinite. With the addition of the relations we get a group G' isomorphic with H, H being merihedrically isomorphic with G.

Then G', or what is the same thing H, may be made isomorphic with a FROBENIUS algebra, which is of use in the applications of the group. A notable application of this kind was made by POINCARÉ.[1]

This application is devoted on the one hand to the study of the linear groups of the periods of the two kinds of integrals of a linear differential equation of order n which is algebraically integrable; and, on the other, to the proof that for every finite group contained in the general linear group of n variables there is such a differential equation. The results are chiefly the following:

494. Theorem. For every group G' there is a system of Fuchsian functions, Abelian integrals of the first kind, such that if $K(z)$ is any such function, and if S is any substitution of G' to which corresponds a linear substitution on z, $\dfrac{\alpha z + \beta}{\gamma z + \delta}$, $(\alpha\delta - \beta\gamma = 1)$, then

$$K\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = K(zS) = K(z) + \omega$$

where ω is called a *period* of $K(z)$.

There are also Abelian integrals of the second kind $P(z, a)$, such that $P(zS, a) = P(z, a) + \phi(a)$, where $\phi(a)$ is the a-derivative of a function of the first kind.

495. Theorem. The genus of the group being q, there are q independent integrals of each kind; all others are expressible linearly in terms of these.

496. Theorem. In the FROBENIUS algebra corresponding to the group let X be $\Sigma \,.\, X_i e_i$, where e_i corresponds to S_i. Then KX means $\Sigma X_i K(z\sigma_i^{-1})$, and ωX means the period of KX corresponding to the period ω of Kz. Then there are three kinds of quadrates in the algebra.

I. Those for which $KX_a =$ constant, for all values of K and any number X_a in the a-th quadrate. In this case $\omega X = 0$ identically for any K and any substitution S, and if $X_a = \Sigma X_{ai} e_i$, there are linear relations among the coefficients X_{ai}. Also $P(z, a) X_a$ is an algebraic function.

II. Those for which KX_a is constant for each K if X_a is properly chosen, so that for any K and any S there is an X_a such that $\omega X_a = 0$.

There is an integral K' whose periods are linear combinations with integral coefficients of X_{ai}; this integral K' combined with K by RIEMANN'S

[1] POINCARÉ 4.

relations gives the coefficients of the periods ω in $\omega X = \Sigma X_{ai}\omega_i$; that is, determines X_{ai}. There are but q such relations independent.

Also $P(z, a) X_a$ is not an algebraic function for this X_a.

III. $K . X_a$ is not constant for certain K's, and any X_a. For any such K we may write $KX_a = G(z)$, then the periods of $G(z)$ being $\omega_1, \omega_2 \ldots . \omega_m$ for the m substitutions of G', if we form the periods of $G(z\sigma)$, we get the same periods ω in another order; a group determinant may be formed from these by letting σ run through G, which must vanish as well as its minors of the first $m - n - 1$ orders.

That there be a rational function of x, y, satisfying a linear equation of order n, it is necessary and sufficient that there are numbers $b_1, b_2 \ldots . b_m$ whose group determinant is of the character above. There is thus always at least one quadrate of the third kind.

497. Theorem. An integral of the first kind, K, *belongs* to a quadrate if $KX =$ constant, for any number X not in this quadrate, but KX is not constant for all numbers in the quadrate.

An integral $P(z, a)$ *belongs* to a quadrate if for all values of X not in this quadrate $P(z, a) X$ is an algebraic function; but for some values of X in the quadrate $P(z, a) X$ is not an algebraic function.

The number of integrals of the first kind belonging to a quadrate of order a^2 is a multiple of a.

Any integral can be separated into integrals each of which belongs to a single quadrate.

498. Theorem. The $2q$ periods of $K(z)$ are subject to a linear transformation by each substitution S of G'. The totality of these linear transformations furnish a linear group isomorphic with H.

The relations between the periods of $P(z, a)$ are found by writing the linear relations between $K(z)$, $K(zS_1)$, $K(zS_2)$, etc., and differentiating them. The derivatives are subject to linear transformations which also generate a group isomorphic with H.

The second group is related to the totality of quadrates of the second kind, the first group to the totality of quadrates of the third kind.

499. Modular Group. This has been studied by means of the commutative algebras.[1]

500. LAURENT[2] has made use of representations of linear substitutions by quadrate numbers or tettarions, to derive several theorems. His processes are briefly indicated below.

Theorem. If $\sigma = \Sigma c_{ij} \lambda_{ij}$, where

$$c_{ii} = 1 \qquad i = 1 \ldots . n \qquad c_{ij} = - c_{ji} \qquad i \neq j$$

then the tettarion $\tau = 2\sigma^{-1} - 1$ represents an *orthogonal* substitution, and the

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[1] J. W. YOUNG 1.

[2] LAURENT 1, 2, 3, 4.

orthogonal group consists of all such substitutions. In this case σ represents a *skew* substitution.

501. **Theorem.** Every orthogonal substitution may be represented by the product of tettarions of the type[1]

$$\omega = \underbrace{\lambda_{11} + \lambda_{22} + \ldots + \lambda_{nn}}_{\lambda_{ii} \text{ and } \lambda_{jj} \text{ absent}} + (\lambda_{ii} + \lambda_{jj}) \cos\phi + (\lambda_{ij} - \lambda_{ji}) \sin\phi$$

502. **Theorem.** Tettarions of the type c and $1 + c(\lambda_{ij} + \lambda_{ji})$ produce tettarions representing *symmetric* substitutions.

503. **Theorem.** Tettarions of the type $1 + \lambda_{ij}$ produce tettarions which represent substitutions with integral coefficients.

504. **Theorem.** If $\tau = \sum\limits_{i,j}^{1\ldots n} a_{ij}\lambda_{ij}$ represents an orthogonal substitution, then $\tau_{pq} = \sum\limits_{i,j}^{1\ldots n} a_{ip}a_{jq}\lambda_{ij}$ gives a new group of linear substitutions. By similar compounding of coefficients of known groups, new groups may be formed.

505. Autonne[2] has applied the theory of matrices to derive theorems relating to linear groups, real, orthogonal, hermitian, and hypohermitian. If

$$\tau = \sum_{i,j}^{1\ldots n} a_{ij}\lambda_{ij} \qquad \breve{\tau} = \Sigma\, a_{ji}\lambda_{ij} \qquad \bar{\tau} = \Sigma\, \bar{a}_{ij}\lambda_{ij}$$

where $\bar{a}_{ij}$ is the conjugate of the ordinary complex number a_{ij}, then τ is *symmetric* if $\breve{\tau} = \tau$; it is *orthogonal* if $\breve{\tau}\tau = 1$; *real* if $\bar{\tau} = \tau$; *unitary* if $\breve{\bar{\tau}}\tau = 1$; *hermitian* if $\breve{\bar{\tau}} = \tau$. In the latter case the *hermitian* form $I.\zeta(\tau)\bar{\zeta} > 0$. [In this expression (τ) acts on ζ as a linear vector operator]. If τ is hermitian there is one and only one hermitian ϕ such that $\phi^2 = \tau$, or $\phi = \tau^{\frac{1}{2}}$.

Theorem. That an n-ary group G can be rendered real and orthogonal by a convenient choice of variables, the following conditions are necessary and sufficient:

(1) G possesses two absolute invariants: a hermitian form $I.\zeta(\tau)\bar{\zeta}$ and an n-ary quadratic form of determinant unity, $P = I.\zeta(\rho)\zeta$.

(2) G having been rendered unitary by being put into the form $\tau^{\frac{1}{2}}G\tau^{-\frac{1}{2}}$, in the transform of P, ρ is unitary.

506. **Theorem.** Every tettarion is the product of a unitary tettarion by a hermitian tettarion.

To put α into such a form we take $\tau^2 = \breve{\bar{\alpha}}\alpha$ and $v = \alpha\tau^{-1}$; then $\alpha = v\tau$.

The literature of bilinear forms furnishes many investigations along these lines.

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[1] Cf. Taber 6, 7, and other papers on matrices.

[2] Autonne 1, 2, 3, 4.

XXVIII. DIFFERENTIAL EQUATIONS.

507. *Pfaff's Equation.* To the solution of the equation

$$X_1\,dx_1 + X_2\,dx_2 + \ldots . + X_m\,dx_m = 0$$

Grassmann[1] applied the methods of the *Ausdehnungslehre.*

508. *La Place's Equation.* This may be written $\nabla^2 u = 0$. It has been treated by quaternionic methods in the case of three variables.[2] Other equations and systems of equations which appear in physics have been handled in analogous ways. The literature of quaternions and vector analysis should be consulted.[3] The full advantage of treating the general operator ∇ as an associative number, would simplify many problems and suggest solutions for cases not yet handled.

509. It is pointed out by Brill[4] that by means of matrices the operator

$$\Delta = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial y^2} + c\frac{\partial^2}{\partial z^2} + f\frac{\partial^2}{\partial y\,\partial z} + g\frac{\partial^2}{\partial z\,\partial x} + h\frac{\partial^2}{\partial x\,\partial y}$$

can be factored into

$$\left\{a\frac{\partial}{\partial x} + (h - ap)\frac{\partial}{\partial y} + (g - aq)\frac{\partial}{\partial z}\right\}\left\{\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial z}\right\}$$

p and q being matrices (tettarions).

Therefore any matrical function of x, y, z which vanishes under the operation of either of these linear operators is a solution of the equation

$$\Delta\,.\,\theta = 0$$

It is obvious that this method is capable of considerable extension.[5]

XXIX. MODULAR SYSTEMS.

510. It is obvious that every multiplication table may be expressed in the form

$$e_i\,e_j - \Sigma\gamma_{ijk}\,e_k = 0$$

If now we consider a domain admitting e_i, e_j, etc., and their products and linear combinations, it is evident that we have a modular system. The expressions e_i .... need not be ordinary algebraic variables, of course; they may be function-signs, for example.

Every modular system may be considered to represent, and may be represented by, an algebra. From this point of view all numbers are qualitative except integers.

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[1] Forsythe 1. Cf. *Ausdehnungslehre*, 1862, §§ 500–527.

[2] Boole 1; Carmichael 1, 2, 3, 4; Brill 2; Graves 1.

[3] Wedderburn 2; Pocklington 1.

[4] Brill 3.

[5] Cf. B. Peirce 2. Same in Appendix I in B. Peirce 3.

XXX. OPERATORS.

511. The use of different abstract algebras in forms which practically make them operators on other entities is quite common in some directions. In such applications the theory demands a consideration of the operands as much as of the operators. As operators they have also certain invariant, covariant, contravariant, etc. operands, so that the invariant theory becomes important.

For example, the algebra of nonions plays a very important part in quaternions as the theory of the linear vector operator.[1]

512. *Invariants of Quantics.* The formulæ and methods of quaternions have been applied to the study of the invariants of the orthogonal transformations of ternary and quaternary quantics.[2] If ξ is a vector, then $q\xi q^{-1}$ is an orthogonal transformation of ξ, q being any quaternion of non-vanishing tensor. Every vector or power of a vector or products of powers of vectors furnishes a pseudo-invariant. Orthogonal ternary invariants are then those functions of vectors which are mere scalars, the list being as follows:

$$T^2\rho \quad T^2\alpha \quad S\alpha\beta \quad S\rho\alpha\beta \quad S\alpha\beta\gamma$$

In these, α, β, etc. are practically different *nablas* operating on ρ, so that we understand by $S \,.\, \alpha\beta\gamma$ substantially what is also written $S \,.\, \nabla_1\nabla_2\nabla_3$. The formulæ of quaternions become thus applicable to these symbolic operators, yielding reductions, syzygies, etc. For example, the syzygies

$$S\alpha\beta\, S\gamma\delta\varepsilon - S\beta\gamma\, S\alpha\delta\varepsilon + S\beta\delta\, S\alpha\gamma\varepsilon - S\beta\varepsilon\, S\alpha\gamma\delta = 0$$
$$S\alpha\rho\, S\beta\gamma\delta - S\beta\rho\, S\alpha\gamma\delta + S\gamma\rho\, S\alpha\beta\delta - S\delta\rho\, S\alpha\beta\gamma = 0$$

This amounts, of course, to a new interpretation of ARONHOLD's notation, and the process may readily be generalized to n dimensions by introducing the forms $I\alpha\beta$, $I\rho\rho$, $I\alpha\rho$, and the like.

513. *Differential Operators.* The differential operators occurring in continuous group-theory are associative, hence generate an associative algebra (usually infinite in dimensions). Groups of such operators are groups in the algebras they define, and their theory may be considered to be a chapter on group-theory of infinite algebras. The whole subject of infinite algebras is undeveloped. The iterative calculus, the calculus of functional equations, and the calculus of linear operations are closely connected with the subject of this memoir.[3]

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[1] See references under Nonions, previously given.

[2] MCMAHON 1; SHAW 14.

[3] PINCHERLE 2, 3; LÉMERAY 1, 2, 3; LEAU 1. The literature of this subject should be consulted.

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